

MULTI-TIME LAGRANGIAN 1-FORMS FOR FAMILIES OF BÄCKLUND TRANSFORMATIONS. TODA-TYPE SYSTEMS

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ABSTRACT. General Lagrangian theory of discrete one-dimensional integrable systems is illustrated by a detailed study of Bäcklund transformations for Toda-type systems. Commutativity of Bäcklund transformations is shown to be equivalent to consistency of the system of discrete multi-time Euler-Lagrange equations. The precise meaning of the commutativity in the periodic case, when all maps are double-valued, is established. It is shown that gluing of different branches is governed by the so called superposition formulas. The closure relation for the multi-time Lagrangian 1-form on solutions of the variational equations is proved for all Toda-type systems. Superposition formulas are instrumental for this proof. The closure relation was previously shown to be equivalent to the spectrality property of Bäcklund transformations, i.e., to the fact that the derivative of the Lagrangian with respect to the spectral parameter is a common integral of motion of the family of Bäcklund transformations. We relate this integral of motion to the monodromy matrix of the zero curvature representation which is derived directly from equations of motion in an algorithmic way. This serves as a further evidence in favor of the idea that Bäcklund transformations serve as zero curvature representations for themselves.

1. INTRODUCTION

The present paper can be considered as an extended illustration of the general Lagrangian theory of discrete integrable systems of classical mechanics, developed in [19]. This development was prompted by an example of the discrete time Calogero-Moser system studied in [22], and belongs to the line of research on variational formulation of more general discrete integrable systems, initiated by Lobb and Nijhoff in [12].

The notion of integrability of discrete systems, lying at the basis of this development, is that of the multidimensional consistency. This understanding of integrability of discrete systems has been a major breakthrough [6], [15], and stimulated an impressive activity boost in the area, cf. [7]. According to the concept of multi-dimensional consistency, integrable d -dimensional systems can be imposed in a consistent way on all d -dimensional sublattices of a lattice \mathbb{Z}^m of arbitrary dimension. This means that the resulting multi-dimensional system possesses solutions whose restrictions to any d -dimensional sublattice are generic solutions of the corresponding two-dimensional system. In the case $d = 1$ the concept of multi-dimensional consistency is more or less synonymous with the idea of integrability as commutativity which has been advocated by Veselov [21]. In the important

case of discrete integrable systems of dimension $d = 2$, this approach led to classification results [2] (ABS list) which turned out to be rather influential.

The Lagrangian aspects of the theory were, as mentioned above, pushed forward in [12]. They observed that the value of the action functional for ABS equations remains invariant under local changes of the underlying quad-surface, and suggested to consider this as a defining feature of integrability. Their results, found on the case-by-case basis for some equations of the ABS list, have been extended to the whole list and given a more conceptual proof in [8], and have been subsequently generalized in various directions: for multi-field two-dimensional systems [13], [4], for asymmetric two-dimensional systems [10], for dKP, the fundamental three-dimensional discrete integrable system [14], and for the above mentioned example of one-dimensional integrable systems [22].

General Lagrangian theory of discrete one-dimensional integrable systems has been developed in [19]. We give a short account of this theory in Section 2. It turns out that the most significant class of examples is given by *Bäcklund transformations*, i.e., one-parameter families of commuting symplectic maps. In the main body of this paper, Sections 4–9, we work out the relevant results for Bäcklund transformations for integrable systems of the Toda type. The list of systems under consideration is given, for convenience of the reader, in Section 3. Our main results are the following.

- (1) Although Bäcklund transformations for the Toda lattice are very well studied (cf. [11] and quotations therein), there is one aspect which was not sufficiently dealt with in the existing literature. Usually one considers these systems under periodic boundary conditions, which yields *multi-valuedness* of the corresponding maps. A general discussion of commutativity of multi-valued maps (correspondences) is contained in [21]. However, its applicability to Bäcklund transformations of Toda-like systems, in particular, the choice of branches ensuring commutativity of such maps seems to be a completely open problem. We give here a complete solution of this problem. The key ingredient are the so called *superposition formulas*, which enable us to precisely describe the branching behavior on the level of local algebraic relations.
- (2) The main feature of the Lagrangian theory of discrete integrable systems is the so called *closure relation*, which expresses the fact that the Lagrangian form on the multi-dimensional space of independent variables is closed on solutions of variational equations. In the case of Bäcklund transformations, it was shown in [19] that the closure relation is equivalent to the so called *spectrality property* introduced by Sklyanin and Kuznetsov in [11], which had, up to now, a somewhat mysterious reputation. We prove spectrality (and thus the closure relation) for all systems of the Toda type. Superposition formulas turn out to be of a crucial importance also for this aim.
- (3) In [19], an idea was pushed forward that Lax representations for Bäcklund transformations are already encoded in the equations of motion themselves. This is a one-dimensional counterpart of an analogous idea for two-dimensional systems, which was one of the main breakthroughs of [6], [15]. Here, we support this idea by an algorithmic derivation of transition and monodromy matrices for all Toda-type integrable systems. We think that a great portion of a mystic flair still enjoyed by integrable systems gets herewith a rational and ultimately simple explanation.

2. GENERAL THEORY OF DISCRETE MULTI-TIME EULER-LAGRANGE EQUATIONS

We will now recall the main positions of the Lagrangian theory of discrete one-dimensional integrable systems, developed in [19], in application to *Bäcklund transformations*, i.e., to one-parameter families of commuting symplectic maps. Suppose that such a symplectic map $F_\lambda : (x, p) \mapsto (\tilde{x}, \tilde{p})$, depending on the parameter λ , admits a generating function Λ :

$$F_\lambda : p = -\frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial x}, \quad \tilde{p} = \frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial \tilde{x}}. \quad (1)$$

Here, the first equation should be (at least locally) solvable for \tilde{x} , i.e., the matrices of the mixed second order partial derivatives of the Lagrange function Λ should be non-degenerate, $\det(\partial^2 \Lambda / \partial x \partial \tilde{x}) \neq 0$. When considering a second such map, say F_μ , we will denote its action by a hat:

$$F_\mu : p = -\frac{\partial \Lambda(x, \hat{x}; \mu)}{\partial x}, \quad \hat{p} = \frac{\partial \Lambda(x, \hat{x}; \mu)}{\partial \hat{x}}. \quad (2)$$

We assume that

$$F_\lambda \circ F_\mu = F_\mu \circ F_\lambda.$$

As a consequence, the following equations, called corner equations, are obtained by eliminating p from (1), (2) at the four vertices of a square on Fig. 1:

$$\frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial x} - \frac{\partial \Lambda(x, \hat{x}; \mu)}{\partial x} = 0, \quad (E)$$

$$\frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial \tilde{x}} + \frac{\partial \Lambda(\tilde{x}, \hat{x}; \mu)}{\partial \tilde{x}} = 0, \quad (E_1)$$

$$\frac{\partial \Lambda(x, \hat{x}; \mu)}{\partial \hat{x}} + \frac{\partial \Lambda(\hat{x}, \tilde{x}; \lambda)}{\partial \hat{x}} = 0, \quad (E_2)$$

and

$$\frac{\partial \Lambda(\hat{x}, \tilde{x}; \lambda)}{\partial \tilde{x}} - \frac{\partial \Lambda(\tilde{x}, \hat{x}; \mu)}{\partial \tilde{x}} = 0. \quad (E_{12})$$

These equations admit the following variational interpretation. We define the *discrete multi-time Lagrangian 1-form* for a family of Bäcklund transformations as a discrete 1-form whose values on the (directed) edges of \mathbb{Z}^2 are given by $\Lambda(x, \tilde{x}; \lambda)$, resp. $\Lambda(x, \hat{x}; \mu)$. A generalization to \mathbb{Z}^m with any $m \geq 2$ is straightforward. We look for functions $x : \mathbb{Z}^m \rightarrow \mathbb{R}$ delivering critical points for the action along *any* discrete curve in \mathbb{Z}^m . Then equations (E)–(E₁₂) are nothing but *multi-time Euler-Lagrange equations* for this variational problem; see [19].

Consistency of the system of multi-time Euler-Lagrange equations (E)–(E₁₂) should be understood as follows: start with the fields x, \tilde{x}, \hat{x} satisfying the corner equation (E). Then each of the corner equations (E₁), (E₂) can be solved for $\hat{\tilde{x}}$. Thus, we obtain two alternative values for the latter field. Consistency takes place if these values coincide identically (with respect to the initial data), and, moreover, if the resulting field $\hat{\tilde{x}}$ satisfies the corner equation (E₁₂). This is equivalent to commutativity of F_λ and F_μ . See Fig. 1.

We mention that the standard single-time Euler-Lagrange equations for the maps F_λ ,

$$\frac{\partial \Lambda(\underline{x}, x; \lambda)}{\partial x} + \frac{\partial \Lambda(x, \tilde{x}; \lambda)}{\partial x} = 0,$$

are a consequence of equation (E) and (the downshifted version of) equation (E₁).

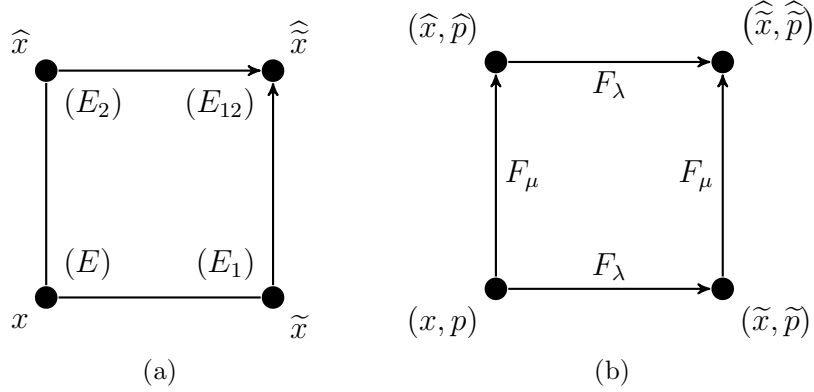


FIGURE 1. Consistency of multi-time Euler-Lagrange equations: (a) Start with data x, \tilde{x}, \hat{x} related by corner equation (E) ; solve corner equations (E_1) and (E_2) for $\hat{\tilde{x}}$; consistency means that the two values of $\hat{\tilde{x}}$ coincide identically and satisfy corner equation (E_{12}) . (b) Maps F_λ and F_μ commute.

As shown in [19], on solutions of discrete multi-time Euler-Lagrange equations, we have:

$$\Lambda(x, \tilde{x}; \lambda) + \Lambda(\tilde{x}, \hat{\tilde{x}}; \mu) - \Lambda(x, \hat{x}; \mu) - \Lambda(\hat{x}, \hat{\tilde{x}}; \lambda) = \ell(\lambda, \mu) = \text{const.} \quad (3)$$

Moreover, $\ell(\lambda, \mu) = 0$, i.e., the discrete multi-time Lagrangian 1-form is closed on solutions, if and only if $\partial\Lambda(x, \tilde{x}; \lambda)/\partial\lambda$ is a common integral of motion for all F_μ . The latter property is a re-formulation of the mysterious “spectrality property” of Bäcklund transformations discovered by Kuznetsov and Sklyanin [11].

3. TODA-TYPE SYSTEMS AND THEIR TIME DISCRETIZATIONS

We will illustrate the above concepts with an important and representative set of examples, namely, Bäcklund transformations for Toda-type systems. The latter term is used to denote integrable lattice equations of the general form

$$\ddot{x}_k = r(\dot{x}_k)(f(x_{k+1} - x_k) - f(x_k - x_{k-1})).$$

The integrable discretizations [18] are of the form

$$g(\tilde{x}_k - x_k; h) - g(x_k - \underline{x}_k; h) = f(\underline{x}_{k+1} - x_k; h) - f(x_k - \tilde{x}_{k-1}; h),$$

with h being an arbitrary parameter (time step). It is this parameter (or its inverse) which will play the role of the Bäcklund parameter λ in all our examples. The list of examples includes:

- The original (exponential) *Toda lattice*:

$$\ddot{x}_k = e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}}, \quad (4)$$

with a discrete time counterpart

$$e^{\tilde{x}_k - x_k} - e^{x_k - \underline{x}_k} = h^2 (e^{\underline{x}_{k+1} - x_k} - e^{x_k - \tilde{x}_{k-1}}). \quad (5)$$

- *Dual Toda lattice*:

$$\ddot{x}_k = \dot{x}_k (x_{k+1} - 2x_k + x_{k-1}), \quad (6)$$

with a discrete time counterpart

$$\frac{\tilde{x}_k - x_k}{x_k - \underline{x}_k} = \frac{1 + h(\underline{x}_{k+1} - x_k)}{1 + h(x_k - \tilde{x}_{k-1})}. \quad (7)$$

- *Modified Toda lattice:*

$$\ddot{x}_k = \dot{x}_k (e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}}), \quad (8)$$

with a discrete time counterpart

$$\frac{e^{\tilde{x}_k - x_k} - 1}{e^{x_k - \underline{x}_k} - 1} = \frac{1 + h e^{\underline{x}_{k+1} - x_k}}{1 + h e^{x_k - \tilde{x}_{k-1}}}. \quad (9)$$

- *Symmetric rational additive Toda-type system:*

$$\ddot{x}_k = -\dot{x}_k^2 \left(\frac{1}{x_{k+1} - x_k} - \frac{1}{x_k - x_{k-1}} \right), \quad (10)$$

with a discrete time counterpart

$$\frac{1}{\tilde{x}_k - x_k} - \frac{1}{x_k - \underline{x}_k} = \frac{1}{\underline{x}_{k+1} - x_k} - \frac{1}{x_k - \tilde{x}_{k-1}}. \quad (11)$$

- *Symmetric rational multiplicative Toda-type system:*

$$\ddot{x}_k = -(\dot{x}_k^2 - 1) \left(\frac{1}{x_{k+1} - x_k} - \frac{1}{x_k - x_{k-1}} \right), \quad (12)$$

with a discrete time counterpart

$$\frac{(\tilde{x}_k - x_k + h)}{(\tilde{x}_k - x_k - h)} \cdot \frac{(x_k - \underline{x}_k - h)}{(x_k - \underline{x}_k + h)} = \frac{(\underline{x}_{k+1} - x_k + h)}{(\underline{x}_{k+1} - x_k - h)} \cdot \frac{(x_k - \tilde{x}_{k-1} - h)}{(x_k - \tilde{x}_{k-1} + h)}. \quad (13)$$

- *Symmetric hyperbolic multiplicative Toda-type system:*

$$\ddot{x}_k = -(\dot{x}_k^2 - 1) (\coth(x_{k+1} - x_k) - \coth(x_k - x_{k-1})), \quad (14)$$

with a discrete time counterpart

$$\frac{\sinh(\tilde{x}_k - x_k + h)}{\sinh(\tilde{x}_k - x_k - h)} \cdot \frac{\sinh(x_k - \underline{x}_k - h)}{\sinh(x_k - \underline{x}_k + h)} = \frac{\sinh(\underline{x}_{k+1} - x_k + h)}{\sinh(\underline{x}_{k+1} - x_k - h)} \cdot \frac{\sinh(x_k - \tilde{x}_{k-1} - h)}{\sinh(x_k - \tilde{x}_{k-1} + h)}. \quad (15)$$

(The names of the last three systems are justified by the appearance of their discrete time versions.)

We consider these systems with finitely many degrees of freedom ($1 \leq k \leq N$). This requires to specify certain boundary conditions. We will consider either periodic boundary conditions (all indices taken mod N , so that $x_0 = x_N$, $x_{N+1} = x_1$), or the so called open-end boundary conditions, which can be imposed in all cases except for the dual Toda lattice and which can be formally achieved by setting $x_0 = \infty$ and $x_{N+1} = -\infty$. For the dual Toda lattice (and for the modified Toda lattice, as well), a certain ersatz for the open-end boundary conditions exists. It is achieved by considering the periodic system with $N + 1$ particles enumerated by $0 \leq k \leq N$ and by restricting it to $x_0 = x_{N+1} = 0$. For such a reduction, all results of the present paper can be established, but we do not deal with it in detail, since the resulting systems have a somewhat different flavor of the affine root system $C_N^{(1)}$ rather than the classical root system A_{N-1} . In particular, these systems do not depend on differences $x_{k+1} - x_k$ alone, due to the presence of the boundary

terms depending on x_1 and on x_N , and therefore the total momentum $\sum_{k=1}^N p_k$ is not a conserved quantity for them.

4. BÄCKLUND TRANSFORMATIONS FOR TODA LATTICE

Here we illustrate the main constructions by the well-known example of Bäcklund transformations for the Toda lattice (4). The maps $F_\lambda : \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2N}$ are given by equations of the type (1):

$$F_\lambda : \begin{cases} p_k = \frac{1}{\lambda} (e^{\tilde{x}_k - x_k} - 1) + \lambda e^{x_k - \tilde{x}_{k-1}}, \\ \tilde{p}_k = \frac{1}{\lambda} (e^{\tilde{x}_k - x_k} - 1) + \lambda e^{x_{k+1} - \tilde{x}_k}, \end{cases} \quad (16)$$

cf. [20], [11], [18]. The corresponding Lagrangian is given by

$$\Lambda(x, \tilde{x}; \lambda) = \frac{1}{\lambda} \sum_{k=1}^N (e^{\tilde{x}_k - x_k} - 1 - (\tilde{x}_k - x_k)) - \lambda \sum_{k=1}^N e^{x_{k+1} - \tilde{x}_k}, \quad (17)$$

and the standard single-time Euler-Lagrange equations coincide with (5) with $h = \lambda$.

In the open-end case, we omit the term with $e^{x_1 - \tilde{x}_0}$ from the expression for p_1 , the term with $e^{x_{N+1} - \tilde{x}_N}$ from the expression for \tilde{p}_N , and we let the second sum in (17) extend over $1 \leq k \leq N-1$ only. In this case, the first equations in (16) are uniquely solved for $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N$ (in this order), to give

$$\begin{aligned} e^{\tilde{x}_1 - x_1} &= 1 + \lambda p_1, & e^{\tilde{x}_2 - x_2} &= 1 + \lambda p_2 - \frac{\lambda^2 e^{x_2 - x_1}}{1 + \lambda p_1}, & \dots &, \\ e^{\tilde{x}_N - x_N} &= 1 + \lambda p_N - \frac{\lambda^2 e^{x_N - x_{N-1}}}{1 + \lambda p_{N-1} - \frac{\lambda^2 e^{x_{N-1} - x_{N-2}}}{1 + \lambda p_{N-2} - \dots - \frac{\lambda^2 e^{x_2 - x_1}}{1 + \lambda p_1}}}. \end{aligned}$$

In the periodic case, all $e^{\tilde{x}_k - x_k}$ can be expressed as analogous infinite periodic continued fractions, and are, therefore, *double-valued* functions of (x, p) .

As discussed in the previous section, commutativity of the maps F_λ, F_μ (in the open-end case, when they are well-defined, i.e., single-valued) is equivalent to consistency of the system of corner equations:

$$\frac{1}{\lambda} (e^{\tilde{x}_k - x_k} - 1) + \lambda e^{x_k - \tilde{x}_{k-1}} = \frac{1}{\mu} (e^{\hat{x}_k - x_k} - 1) + \mu e^{x_k - \hat{x}_{k-1}}, \quad (E)$$

$$\frac{1}{\lambda} (e^{\tilde{x}_k - x_k} - 1) + \lambda e^{x_{k+1} - \tilde{x}_k} = \frac{1}{\mu} (e^{\hat{\tilde{x}}_k - \tilde{x}_k} - 1) + \mu e^{\tilde{x}_k - \hat{\tilde{x}}_{k-1}}, \quad (E_1)$$

$$\frac{1}{\mu} (e^{\hat{x}_k - x_k} - 1) + \mu e^{x_{k+1} - \hat{x}_k} = \frac{1}{\lambda} (e^{\hat{\tilde{x}}_k - \hat{x}_k} - 1) + \lambda e^{\hat{x}_k - \hat{\tilde{x}}_{k-1}}, \quad (E_2)$$

$$\frac{1}{\lambda} (e^{\hat{\tilde{x}}_k - \hat{x}_k} - 1) + \lambda e^{\hat{x}_{k+1} - \hat{\tilde{x}}_k} = \frac{1}{\mu} (e^{\hat{\tilde{x}}_k - \hat{\tilde{x}}_k} - 1) + \mu e^{\hat{\tilde{x}}_{k+1} - \hat{\tilde{x}}_k}. \quad (E_{12})$$

We have to clarify the meaning of the both notions (commutativity of F_λ, F_μ and consistency of corner equations) in the periodic case. To do this, we prove the following statement.

Theorem 1. Suppose that the fields x, \tilde{x}, \hat{x} satisfy corner equations (E). Define the fields $\hat{\tilde{x}}$ by any of the following two formulas, which are equivalent by virtue of (E):

$$\frac{1}{\lambda} \left(e^{\hat{\tilde{x}}_k - \hat{x}_k} - 1 \right) - \frac{1}{\mu} \left(e^{\hat{\tilde{x}}_k - \tilde{x}_k} - 1 \right) + \lambda e^{x_{k+1} - \tilde{x}_k} - \mu e^{x_{k+1} - \hat{x}_k} = 0, \quad (S1)$$

$$\frac{1}{\lambda} \left(e^{\tilde{x}_{k+1} - x_{k+1}} - 1 \right) - \frac{1}{\mu} \left(e^{\tilde{x}_{k+1} - x_{k+1}} - 1 \right) + \lambda e^{\hat{x}_{k+1} - \hat{\tilde{x}}_k} - \mu e^{\tilde{x}_{k+1} - \hat{\tilde{x}}_k} = 0, \quad (S2)$$

called *superposition formulas*. Then the corner equations (E_1) – (E_{12}) are satisfied, as well.

Proof. First of all, we show that equations (S1) and (S2) are indeed equivalent by virtue of (E). For this, we re-write these equations in algebraically equivalent forms:

$$\frac{\lambda - \mu}{e^{\hat{\tilde{x}}_k - x_{k+1}} - \lambda \mu} = \lambda e^{x_{k+1} - \tilde{x}_k} - \mu e^{x_{k+1} - \hat{x}_k}, \quad (18)$$

and

$$\frac{(\lambda - \mu) e^{x_{k+1} - \hat{\tilde{x}}_k}}{1 - \lambda \mu e^{x_{k+1} - \hat{\tilde{x}}_k}} = \frac{1}{\mu} \left(e^{\tilde{x}_{k+1} - x_{k+1}} - 1 \right) - \frac{1}{\lambda} \left(e^{\tilde{x}_{k+1} - x_{k+1}} - 1 \right), \quad (19)$$

respectively. The left-hand sides of the latter two equations are equal. Thus, their difference coincides with (E).

Second, we show that equations (S1) and (S2) yield (E_1) . (For (E_2) everything is absolutely analogous.) For this aim, we re-write these equations in still other algebraically equivalent forms. Namely, (S1) is equivalent to

$$e^{\hat{\tilde{x}}_k - \tilde{x}_k} = \lambda \mu e^{x_{k+1} - \tilde{x}_k} + \frac{\mu - \lambda}{\mu e^{\tilde{x}_k - \hat{x}_k} - \lambda}, \quad (20)$$

while (S2) with k replaced by $k - 1$ is equivalent to

$$\lambda \mu e^{\tilde{x}_k - \hat{\tilde{x}}_{k-1}} = e^{\tilde{x}_k - x_k} + \frac{\lambda - \mu}{\mu - \lambda e^{\hat{x}_k - \tilde{x}_k}}. \quad (21)$$

An obvious linear combination of these expressions leads to

$$\frac{1}{\mu} e^{\hat{\tilde{x}}_k - \tilde{x}_k} + \mu e^{\tilde{x}_k - \hat{\tilde{x}}_{k-1}} = \lambda e^{x_{k+1} - \tilde{x}_k} + \frac{1}{\lambda} e^{\tilde{x}_k - x_k} + \frac{\lambda - \mu}{\lambda \mu},$$

which is nothing but (E_1) .

Third, we observe that the sum of equations (E), (S1) and (S2) is nothing but the corner equation (E_{12}) . ■

Remark. Observe that each of the equations (S1) and (S2) is a quad-equation with respect to

$$\left(e^{x_{k+1}}, e^{\tilde{x}_k}, e^{\hat{x}_k}, e^{\hat{\tilde{x}}_k} \right), \quad \text{resp.} \quad \left(e^{x_{k+1}}, e^{\tilde{x}_{k+1}}, e^{\hat{x}_{k+1}}, e^{\hat{\tilde{x}}_k} \right),$$

i.e., can be formulated as vanishing of a multi-affine polynomial of the four specified variables. Equations (18) and (19) are then interpreted as the three-leg forms of the original equations, centered at x_{k+1} . Similarly, equations (20) and (21) are the three-leg forms of the original equations, centered at \tilde{x}_k .

This theorem allows us to achieve an exhaustive understanding of the consistency for double-valued Bäcklund transformations. First, suppose that we are given the fields x, \tilde{x}, \hat{x} satisfying the corner equation (E). Each of equations (E_1) , (E_2) produces two values for $\hat{\tilde{x}}$. Then consistency is reflected in the following fact: one of the values for $\hat{\tilde{x}}$ obtained

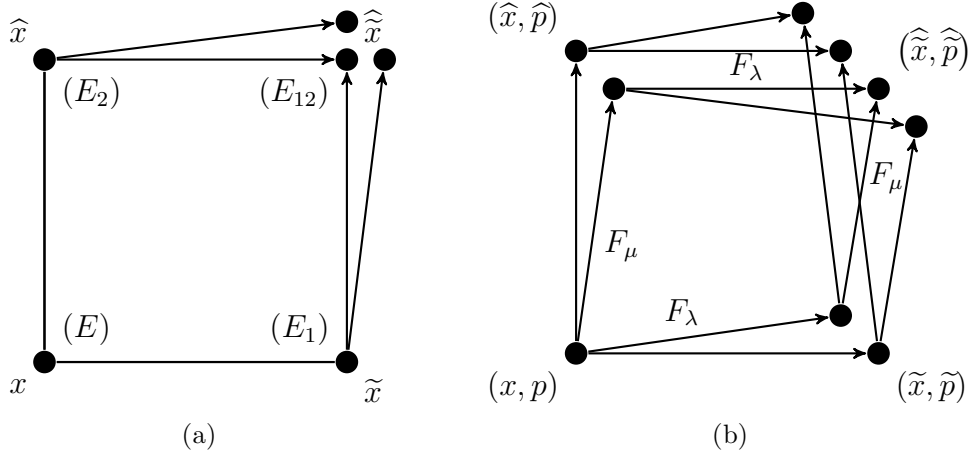


FIGURE 2. Consistency of multi-time Euler-Lagrange equations in the case of periodic boundary conditions: (a) Given the fields x , \tilde{x} , \hat{x} satisfying the corner equation (E) , each of the corner equations (E_1) , (E_2) has two solutions $\hat{\tilde{x}}$. One of the values for $\hat{\tilde{x}}$ obtained from (E_1) coincides with one of the values for $\hat{\tilde{x}}$ obtained from (E_2) . This common value of $\hat{\tilde{x}}$, together with \tilde{x} and \hat{x} , satisfies (E_{12}) . (b) The four branches of $F_\lambda \circ F_\mu$ and the four branches of $F_\mu \circ F_\lambda$ coincide pairwise.

from (E_1) coincides with one of the values for $\hat{\tilde{x}}$ obtained from (E_2) , see Fig. 2(a). Indeed, this common value is nothing but $\hat{\tilde{x}}$ obtained from the superposition formulas $(S1)$, $(S2)$, as in Theorem 1.

The “loose ends” on Fig. 2(a) are best explained by considering the (double-valued) maps F_λ , F_μ , i.e., by working with the variables (x, p) rather than with the variables x alone. Indeed, each of the compositions $F_\lambda \circ F_\mu$ and $F_\mu \circ F_\lambda$ is four-valued. It follows from Theorem 1 that their branches must pairwise coincide, as shown on Fig. 2(b). Indeed, Theorem 1 delivers four possible values for $(\tilde{x}, \hat{x}, \hat{\tilde{x}})$ satisfying all corner equations (E) – (E_{12}) , namely one $\hat{\tilde{x}}$ for each of the four possible combinations of (\tilde{x}, \hat{x}) .

Theorem 2. *The discrete multi-time Lagrangian 1-form is closed on any solution of the corner equations (E) – (E_{12}) .*

Proof. First of all, we show that the closure relation $\ell(\lambda, \mu) = 0$ is equivalent to

$$\sum_{k=1}^N (\hat{\tilde{x}}_k - \tilde{x}_k - \hat{x}_k + x_k) = 0 \quad \Leftrightarrow \quad \prod_{k=1}^N e^{\hat{\tilde{x}}_k - \tilde{x}_k - \hat{x}_k + x_k} = 1. \quad (22)$$

This can be done in two different ways. On one hand, combining $(S1)$, $(S2)$ with (E) , we arrive at the two formulas

$$\frac{1}{\lambda} e^{\tilde{x}_{k+1} - x_{k+1}} - \frac{1}{\mu} e^{\hat{x}_{k+1} - x_{k+1}} - \frac{1}{\lambda} e^{\hat{\tilde{x}}_k - \hat{x}_k} + \frac{1}{\mu} e^{\hat{\tilde{x}}_k - \tilde{x}_k} = 0, \quad (23)$$

$$\lambda e^{x_{k+1} - \tilde{x}_k} - \mu e^{x_{k+1} - \hat{x}_k} - \lambda e^{\hat{x}_{k+1} - \hat{\tilde{x}}_k} + \mu e^{\tilde{x}_{k+1} - \hat{\tilde{x}}_k} = 0. \quad (24)$$

By virtue of these formulas, most of the terms on the left-hand side of (3) with the Lagrange function (17) cancel, leaving us with

$$\ell(\lambda, \mu) = \left(\frac{1}{\lambda} - \frac{1}{\mu} \right) \sum_{k=1}^N (\widehat{x}_k - \widetilde{x}_k - \widehat{x}_k + x_k).$$

For an alternative proof of the fact that $\ell(\lambda, \mu) = 0$ is equivalent to (22), we can refer to the spectrality criterium stating that $\ell(\lambda, \mu) = 0$ is equivalent to $\partial\Lambda(x, \widetilde{x}; \lambda)/\partial\lambda$ being an integral of motion for F_μ . One easily computes:

$$\frac{\partial\Lambda(x, \widetilde{x}; \lambda)}{\partial\lambda} = -\frac{1}{\lambda} \sum_{k=1}^N p_k + \frac{1}{\lambda^2} \sum_{k=1}^N (\widetilde{x}_k - x_k),$$

the first sum on the right-hand side being an obvious integral of motion.

Now the desired result (22) can be derived from the following form of the superposition formula:

$$e^{\widehat{x}_k - \widetilde{x}_k - \widehat{x}_k + x_{k+1}} = \frac{\lambda e^{\widehat{x}_{k+1}} - \mu e^{\widetilde{x}_{k+1}}}{\lambda e^{\widehat{x}_k} - \mu e^{\widetilde{x}_k}}, \quad (25)$$

which is in fact equivalent to either of equations (23), (24). In the periodic case (22) follows directly by multiplying equations (25) for $1 \leq k \leq N$, in the open-end case equation (25) holds true for $1 \leq k \leq N-1$ and has to be supplemented by the boundary counterparts

$$e^{x_1} = \frac{\lambda e^{\widehat{x}_1} - \mu e^{\widetilde{x}_1}}{\lambda - \mu}, \quad e^{\widehat{x}_N - \widetilde{x}_N - \widehat{x}_N} = \frac{\lambda - \mu}{\lambda e^{\widehat{x}_N} - \mu e^{\widetilde{x}_N}}, \quad (26)$$

which are equivalent to (E) for $k=1$, resp. to (E_{12}) for $k=N$. ■

Thus, we have proved that the quantity $\sum_{k=1}^N (\widetilde{x}_k - x_k)$ is an integral of motion for F_μ with an arbitrary μ . Actually, it is not difficult to give a matrix expression for this integral. The participating matrices are nothing but transition matrices of the zero curvature representation for F_λ , but the latter notion is not necessary for establishing the result.

Theorem 3. *Set*

$$L_k(x, p; \lambda) = \begin{pmatrix} 1 + \lambda p_k & -\lambda^2 e^{x_k} \\ e^{-x_k} & 0 \end{pmatrix},$$

and

$$T_N(x, p; \lambda) = L_N(x, p; \lambda) \cdots L_2(x, p; \lambda) L_1(x, p; \lambda).$$

Then in the periodic case the quantity $\prod_{k=1}^N e^{\widetilde{x}_k - x_k}$ is an eigenvalue of $T_N(x, p; \lambda)$, while in the open-end case it is equal to the (11)-entry of $T_N(x, p; \lambda)$.

Proof. We use the following notation for the action of matrices from $GL(2, \mathbb{C})$ on \mathbb{C} by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [z] = \frac{az + b}{cz + d}.$$

With this notation, we can re-write the first equation in (16) as

$$e^{\widetilde{x}_k} = e^{x_k} \left((1 + \lambda p_k) - \lambda^2 e^{x_k - \widetilde{x}_{k-1}} \right) = L_k(x, p; \lambda) [e^{\widetilde{x}_{k-1}}].$$

This is equivalent to saying that

$$L_k(x, p; \lambda) \begin{pmatrix} e^{\widetilde{x}_{k-1}} \\ 1 \end{pmatrix} \sim \begin{pmatrix} e^{\widetilde{x}_k} \\ 1 \end{pmatrix}.$$

The proportionality coefficient is easily determined by comparing the second components of these vectors:

$$L_k(x, p; \lambda) \begin{pmatrix} e^{\tilde{x}_{k-1}} \\ 1 \end{pmatrix} = e^{\tilde{x}_{k-1} - x_k} \begin{pmatrix} e^{\tilde{x}_k} \\ 1 \end{pmatrix}. \quad (27)$$

Now the claim in the periodic case follows immediately, with the corresponding eigenvector of $T_N(x, p; \lambda)$ being $(e^{\tilde{x}_N}, 1)^T$. In the open-end case, equation (27) holds true for $2 \leq k \leq N$, and has to be supplemented by the following two relations:

$$L_1(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-x_1} \begin{pmatrix} e^{\tilde{x}_1} \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\tilde{x}_N} \\ 1 \end{pmatrix} = e^{\tilde{x}_N}.$$

As a consequence,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} T_N(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \prod_{k=1}^N e^{\tilde{x}_k - x_k}. \quad \blacksquare$$

In the subsequent sections, we prove similar results for Bäcklund transformations for all the remaining Toda-type systems. For each of them,

- we find superposition formulas which yield commutativity of Bäcklund transformations, both in the single-valued case of open-end boundary conditions and in the double-valued case of periodic boundary conditions;
- we prove the spectrality property, so that the discrete 1-form \mathcal{L} is closed on solutions of the Euler-Lagrange equations. The proof is based on superposition formulas. This result provides also the existence of a big number of common integrals for the whole family F_λ . Actually, there are sufficiently many common integrals in involution to ensure complete integrability in the Liouville-Arnold sense;
- and we give an expression of the corresponding conserved quantity $\partial\Lambda(x, \tilde{x}; \lambda)/\partial\lambda$ in terms of canonically conjugate variables (x, p) . This is done with the help of the monodromy matrix of the corresponding discrete time zero curvature representation, which is *derived* directly from equations of motion in an unambiguous and algorithmic way. This gives a further support to the idea pushed forward in [19], that Bäcklund transformations can serve as zero curvature representations for themselves.

5. BÄCKLUND TRANSFORMATIONS FOR DUAL TODA LATTICE

Our second example constitute Bäcklund transformations for the dual Toda lattice (6) which are given by equations of the type (1):

$$F_\lambda : \begin{cases} e^{p_k} = (\tilde{x}_k - x_k)(\lambda + x_k - \tilde{x}_{k-1}), \\ e^{\tilde{p}_k} = (\tilde{x}_k - x_k)(\lambda + x_{k+1} - \tilde{x}_k). \end{cases} \quad (28)$$

The corresponding Lagrangian is given by

$$\Lambda(x, \tilde{x}; \lambda) = \sum_{k=1}^N \psi(\tilde{x}_k - x_k) - \sum_{k=1}^N \psi(\lambda + x_{k+1} - \tilde{x}_k),$$

where $\psi(\xi) = \xi \log \xi - \xi$. The standard single-time Euler-Lagrange equations coincide with (7) with $h = \lambda^{-1}$. Recall that for the dual Toda lattice we only consider periodic boundary conditions.

To establish commutativity of the maps F_λ , F_μ , we consider the system of corner equations:

$$(\tilde{x}_k - x_k)(\lambda + x_k - \tilde{x}_{k-1}) = (\hat{x}_k - x_k)(\mu + x_k - \hat{x}_{k-1}), \quad (E)$$

$$(\tilde{x}_k - x_k)(\lambda + x_{k+1} - \tilde{x}_k) = (\hat{x}_k - \tilde{x}_k)(\mu + \tilde{x}_k - \hat{x}_{k-1}), \quad (E_1)$$

$$(\hat{x}_k - x_k)(\mu + x_{k+1} - \hat{x}_k) = (\hat{x}_k - \hat{x}_k)(\lambda + \hat{x}_k - \hat{x}_{k-1}), \quad (E_2)$$

$$(\hat{x}_k - \hat{x}_k)(\lambda + \hat{x}_{k+1} - \hat{x}_k) = (\hat{x}_k - \tilde{x}_k)(\mu + \tilde{x}_{k+1} - \hat{x}_k). \quad (E_{12})$$

Theorem 4. *Suppose that the fields x , \tilde{x} , \hat{x} satisfy corner equations (E). Define the fields $\hat{\tilde{x}}$ by any of the following two superposition formulas, which are equivalent by virtue of (E):*

$$(\hat{x}_k - \hat{x}_k)(\lambda + x_{k+1} - \tilde{x}_k) = (\hat{x}_k - \tilde{x}_k)(\mu + x_{k+1} - \hat{x}_k), \quad (S1)$$

$$(\tilde{x}_{k+1} - x_{k+1})(\lambda + \hat{x}_{k+1} - \hat{\tilde{x}}_k) = (\hat{x}_{k+1} - x_{k+1})(\mu + \tilde{x}_{k+1} - \hat{\tilde{x}}_k). \quad (S2)$$

Then the corner equations (E₁)–(E₁₂) are satisfied, as well.

Proof. Each of the equations (S1) and (S2) is a quad-equation with respect to

$$(x_{k+1}, \tilde{x}_k, \hat{x}_k, \hat{\tilde{x}}_k), \quad \text{resp.} \quad (x_{k+1}, \tilde{x}_{k+1}, \hat{x}_{k+1}, \hat{\tilde{x}}_k).$$

The three-leg forms of these equations, centered at x_{k+1} , are:

$$\frac{\lambda + x_{k+1} - \hat{\tilde{x}}_k}{\mu + x_{k+1} - \hat{x}_k} = \frac{\lambda + x_{k+1} - \tilde{x}_k}{\mu + x_{k+1} - \hat{x}_k},$$

and

$$\frac{\lambda + x_{k+1} - \hat{\tilde{x}}_k}{\mu + x_{k+1} - \hat{x}_k} = \frac{\hat{x}_{k+1} - x_{k+1}}{\tilde{x}_{k+1} - x_{k+1}},$$

respectively. Their quotient coincides with (E).

The three-leg forms of equations (S1) and (S2), centered at \tilde{x}_k , are:

$$\frac{\hat{x}_k - \tilde{x}_k}{\lambda - \mu + \hat{x}_k - \tilde{x}_k} = \frac{\hat{\tilde{x}}_k - \tilde{x}_k}{\lambda + x_{k+1} - \tilde{x}_k},$$

and

$$\frac{\hat{x}_{k+1} - \tilde{x}_{k+1}}{\lambda - \mu + \hat{x}_{k+1} - \tilde{x}_{k+1}} = \frac{\tilde{x}_{k+1} - x_{k+1}}{\mu + \tilde{x}_{k+1} - \hat{\tilde{x}}_k},$$

respectively. The quotient of these two equations (the second with the downshifted index k) coincides with (E₁).

The three-leg forms of superposition formulas (S1) and (S2), centered at $\hat{\tilde{x}}_k$, are:

$$\frac{\lambda + x_{k+1} - \hat{\tilde{x}}_k}{\mu + x_{k+1} - \hat{x}_k} = \frac{\hat{x}_k - \tilde{x}_k}{\hat{\tilde{x}}_k - \hat{x}_k},$$

and

$$\frac{\lambda + x_{k+1} - \hat{\tilde{x}}_k}{\mu + x_{k+1} - \hat{x}_k} = \frac{\lambda + \hat{x}_{k+1} - \hat{\tilde{x}}_k}{\mu + \tilde{x}_{k+1} - \hat{\tilde{x}}_k},$$

respectively. The quotient of these two equations coincides with (E₁₂). ■

Theorem 5. *The discrete multi-time Lagrangian 1-form is closed on any solution of the corner equations (E)–(E₁₂).*

Proof. With the help of the spectrality criterium, we see that the claim of the theorem is equivalent to

$$\prod_{k=1}^N (\lambda + \widehat{x}_{k+1} - \widehat{\widetilde{x}}_k) = \prod_{k=1}^N (\lambda + x_{k+1} - \widetilde{x}_k). \quad (29)$$

To prove this relation, we observe that superposition formulas (S1) and (S2) admit the following further equivalent formulations:

$$(\lambda - \mu)(\lambda + x_{k+1} - \widetilde{x}_k) = (\lambda - \mu + \widehat{x}_k - \widetilde{x}_k)(\lambda + x_{k+1} - \widehat{\widetilde{x}}_k),$$

and

$$(\lambda - \mu)(\lambda + \widehat{x}_{k+1} - \widehat{\widetilde{x}}_k) = (\lambda - \mu + \widehat{x}_{k+1} - \widetilde{x}_{k+1})(\lambda + x_{k+1} - \widehat{\widetilde{x}}_k),$$

respectively. There follows:

$$\frac{\lambda + \widehat{x}_{k+1} - \widehat{\widetilde{x}}_k}{\lambda + x_{k+1} - \widetilde{x}_k} = \frac{\mu - \lambda + \widetilde{x}_{k+1} - \widehat{x}_{k+1}}{\mu - \lambda + \widetilde{x}_k - \widehat{x}_k}.$$

Under the periodic boundary conditions, this formula yields (29). ■

Theorem 6. *Set*

$$L_k(x, p; \lambda) = \begin{pmatrix} -x_k & x_k(\lambda + x_k) + e^{p_k} \\ -1 & \lambda + x_k \end{pmatrix},$$

and

$$T_N(x, p; \lambda) = L_N(x, p; \lambda) \cdots L_2(x, p; \lambda) L_1(x, p; \lambda).$$

Then, under the periodic boundary conditions, the quantity $\prod_{k=1}^N (\lambda + x_{k+1} - \widetilde{x}_k)$ is an eigenvalue of $T_N(x, p; \lambda)$.

Proof. We can re-write the first equation in (28) as

$$\widetilde{x}_k = x_k + \frac{e^{p_k}}{\lambda + x_k - \widetilde{x}_{k-1}} = \frac{(\lambda + x_k)x_k + e^{p_k} - x_k \widetilde{x}_{k-1}}{\lambda + x_k - \widetilde{x}_{k-1}} = L_k(x, p; \lambda)[\widetilde{x}_{k-1}].$$

This is equivalent to

$$L_k(x, p; \lambda) \begin{pmatrix} \widetilde{x}_{k-1} \\ 1 \end{pmatrix} \sim \begin{pmatrix} \widetilde{x}_k \\ 1 \end{pmatrix}.$$

The proportionality coefficient is determined by comparing the second components of these two vectors:

$$L_k(x, p; \lambda) \begin{pmatrix} \widetilde{x}_{k-1} \\ 1 \end{pmatrix} = (\lambda + x_k - \widetilde{x}_{k-1}) \begin{pmatrix} \widetilde{x}_k \\ 1 \end{pmatrix}.$$

Now the claim follows immediately from the periodic boundary conditions, with the corresponding eigenvector of $T_N(x, p; \lambda)$ being $(\widetilde{x}_N, 1)^T$. ■

6. BÄCKLUND TRANSFORMATIONS FOR MODIFIED TODA LATTICE

Our third example constitute Bäcklund transformations for the modified Toda lattice (8) which are given by equations of the type (1):

$$F_\lambda : \begin{cases} e^{p_k} = (e^{\tilde{x}_k - x_k} - 1) (\lambda + e^{x_k - \tilde{x}_{k-1}}), \\ e^{\tilde{p}_k} = (e^{\tilde{x}_k - x_k} - 1) (\lambda + e^{x_{k+1} - \tilde{x}_k}). \end{cases} \quad (30)$$

Like for the standard Toda lattice, in the open-end case the first equations in (30) are uniquely solved for $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N$ (in this order), and in the periodic case \tilde{x}_k are double-valued functions of (x, p) .

The corresponding Lagrangian is given in the periodic case by

$$\Lambda(x, \tilde{x}; \lambda) = \sum_{k=1}^N \psi(\tilde{x}_k - x_k; -1) - \sum_{k=1}^N \psi(x_{k+1} - \tilde{x}_k; \lambda),$$

and in the open-end case by

$$\Lambda(x, \tilde{x}; \lambda) = \sum_{k=1}^N \psi(\tilde{x}_k - x_k; -1) - \sum_{k=1}^{N-1} \psi(x_{k+1} - \tilde{x}_k; \lambda) + (\tilde{x}_N - x_1) \log \lambda,$$

where

$$\psi(\xi; \lambda) = \int_0^\xi \log(e^\eta + \lambda) d\eta.$$

The standard single-time Euler-Lagrange equations are (9) with $h = \lambda^{-1}$.

To establish commutativity of the maps F_λ, F_μ , we consider the system of corner equations:

$$(e^{\tilde{x}_k - x_k} - 1) (\lambda + e^{x_k - \tilde{x}_{k-1}}) = (e^{\hat{x}_k - x_k} - 1) (\mu + e^{x_k - \hat{x}_{k-1}}), \quad (E)$$

$$(e^{\tilde{x}_k - x_k} - 1) (\lambda + e^{x_{k+1} - \tilde{x}_k}) = (e^{\hat{x}_k - \tilde{x}_k} - 1) (\mu + e^{\tilde{x}_k - \hat{x}_{k-1}}), \quad (E_1)$$

$$(e^{\hat{x}_k - x_k} - 1) (\mu + e^{x_{k+1} - \hat{x}_k}) = (e^{\hat{x}_k - \tilde{x}_k} - 1) (\lambda + e^{\hat{x}_k - \tilde{x}_{k-1}}), \quad (E_2)$$

$$(e^{\hat{x}_k - \hat{x}_k} - 1) (\lambda + e^{\hat{x}_{k+1} - \hat{x}_k}) = (e^{\hat{x}_k - \tilde{x}_k} - 1) (\mu + e^{\tilde{x}_{k+1} - \hat{x}_k}). \quad (E_{12})$$

Theorem 7. *Suppose that the fields x, \tilde{x}, \hat{x} satisfy corner equations (E). Define the fields $\hat{\tilde{x}}$ by any of the following two superposition formulas, which are equivalent by virtue of (E):*

$$(e^{\hat{\tilde{x}}_k - \hat{x}_k} - 1) (\lambda + e^{x_{k+1} - \tilde{x}_k}) = (e^{\hat{\tilde{x}}_k - \tilde{x}_k} - 1) (\mu + e^{x_{k+1} - \hat{x}_k}), \quad (S1)$$

$$(e^{\tilde{x}_{k+1} - x_{k+1}} - 1) (\lambda + e^{\hat{x}_{k+1} - \hat{x}_k}) = (e^{\hat{x}_{k+1} - x_{k+1}} - 1) (\mu + e^{\tilde{x}_{k+1} - \hat{x}_k}). \quad (S2)$$

Then the corner equations $(E_1)-(E_{12})$ are satisfied, as well.

Proof. Each of the equations (S1) and (S2) is a quad-equation with respect to

$$(e^{x_{k+1}}, e^{\tilde{x}_k}, e^{\hat{x}_k}, e^{\hat{\tilde{x}}_k}), \quad \text{resp.} \quad (e^{x_{k+1}}, e^{\tilde{x}_{k+1}}, e^{\hat{x}_{k+1}}, e^{\hat{\tilde{x}}_k}).$$

The three-leg forms of equations (S1) and (S2), centered at x_{k+1} , are:

$$\frac{\lambda + e^{x_{k+1} - \hat{\tilde{x}}_k}}{\mu + e^{x_{k+1} - \hat{x}_k}} = \frac{\lambda + e^{x_{k+1} - \tilde{x}_k}}{\mu + e^{x_{k+1} - \hat{x}_k}},$$

and

$$\frac{\lambda + e^{x_{k+1} - \widehat{x}_k}}{\mu + e^{x_{k+1} - \widehat{x}_k}} = \frac{e^{\widehat{x}_{k+1} - x_{k+1}} - 1}{e^{\widehat{x}_{k+1} - x_{k+1}} - 1},$$

respectively. Their quotient coincides with (E) .

The three-leg forms of equations $(S1)$ and $(S2)$, centered at \widetilde{x}_k , are:

$$\frac{e^{\widehat{x}_k - \widetilde{x}_k} - 1}{\lambda - \mu e^{\widehat{x}_k - \widetilde{x}_k}} = \frac{e^{\widetilde{x}_k - \widetilde{x}_k} - 1}{\lambda + e^{x_{k+1} - \widetilde{x}_k}},$$

and

$$\frac{e^{\widehat{x}_{k+1} - \widetilde{x}_{k+1}} - 1}{\lambda - \mu e^{\widehat{x}_{k+1} - \widetilde{x}_{k+1}}} = \frac{e^{\widetilde{x}_{k+1} - x_{k+1}} - 1}{\mu + e^{\widetilde{x}_{k+1} - \widehat{x}_k}},$$

respectively. The quotient of these two equations (the second with the downshifted index k) coincides with (E_1) .

The three-leg forms of superposition formulas $(S1)$ and $(S2)$, centered at \widehat{x}_k , are:

$$\frac{\lambda + e^{x_{k+1} - \widehat{x}_k}}{\mu + e^{x_{k+1} - \widehat{x}_k}} = \frac{e^{\widehat{x}_k - \widetilde{x}_k} - 1}{e^{\widehat{x}_k - \widetilde{x}_k} - 1},$$

and

$$\frac{\lambda + e^{x_{k+1} - \widehat{x}_k}}{\mu + e^{x_{k+1} - \widehat{x}_k}} = \frac{\lambda + e^{\widehat{x}_{k+1} - \widehat{x}_k}}{\mu + e^{\widetilde{x}_{k+1} - \widehat{x}_k}},$$

respectively. The quotient of these two equations coincides with (E_{12}) . ■

Theorem 8. *The discrete multi-time Lagrangian 1-form is closed on any solution of the corner equations (E) – (E_{12}) .*

Proof. With the help of the spectrality criterium, we see that the claim of the theorem in the periodic case is equivalent to

$$P(x, \widetilde{x}) = P(\widehat{x}, \widehat{x}), \quad (31)$$

where in the periodic case

$$P(x, \widetilde{x}) = \prod_{k=1}^N (1 + \lambda e^{\widetilde{x}_k - x_{k+1}}),$$

while in the open-end case

$$P(x, \widetilde{x}) = e^{\widetilde{x}_N - x_1} \prod_{k=1}^{N-1} (1 + \lambda e^{\widetilde{x}_k - x_{k+1}}).$$

To prove this relation, we observe that superposition formulas $(S1)$ and $(S2)$ admit the following further equivalent formulations:

$$(\lambda - \mu)(1 + \lambda e^{\widetilde{x}_k - x_{k+1}}) = (\lambda e^{\widetilde{x}_k - \widehat{x}_k} - \mu)(1 + \lambda e^{\widehat{x}_k - x_{k+1}}),$$

and

$$(\lambda - \mu)(1 + \lambda e^{\widehat{x}_k - \widehat{x}_{k+1}}) = (\lambda e^{\widetilde{x}_{k+1} - \widehat{x}_{k+1}} - \mu)(1 + \lambda e^{\widehat{x}_k - x_{k+1}}),$$

respectively. There follows:

$$\frac{1 + \lambda e^{\hat{x}_k - \hat{x}_{k+1}}}{1 + \lambda e^{\tilde{x}_k - x_{k+1}}} = \frac{\lambda e^{\tilde{x}_{k+1} - \hat{x}_{k+1}} - \mu}{\lambda e^{\tilde{x}_k - \hat{x}_k} - \mu}. \quad (32)$$

In the periodic case this formula yields (31). In the open-end case equation (32) holds true for $1 \leq k \leq N-1$, and has to be supplemented with

$$\frac{\lambda e^{\tilde{x}_1 - \hat{x}_1} - \mu}{\lambda - \mu} = e^{x_1 - \hat{x}_1}, \quad \frac{\lambda - \mu}{\lambda e^{\tilde{x}_N - \hat{x}_N} - \mu} = e^{\hat{x}_N - \tilde{x}_N}, \quad (33)$$

which are equivalent to (E) for $k=1$, resp. to (E_{12}) for $k=N$. Product of equations (32) with $1 \leq k \leq N-1$ and equations (33) yields the claim of the theorem in the open-end case, as well. ■

Theorem 9. *Set*

$$L_k(x, p; \lambda) = \begin{pmatrix} \lambda + e^{p_k} & e^{x_k} \\ \lambda e^{-x_k} & 1 \end{pmatrix},$$

and

$$T_N(x, p; \lambda) = L_N(x, p; \lambda) \cdots L_2(x, p; \lambda) L_1(x, p; \lambda).$$

Then in the periodic case the quantity $P(x, \tilde{x})$ from (6) is an eigenvalue of $T_N(x, p; \lambda)$, while in the open-end case its counterpart from (6) is equal, up to the factor λ , to the (11)-entry of $T_N(x, p; \lambda)$.

Proof. We can re-write the first equation in (30) as

$$e^{\tilde{x}_k} = e^{x_k} \left(1 + \frac{e^{p_k}}{\lambda + e^{x_k - \tilde{x}_{k-1}}} \right) = \frac{(\lambda + e^{p_k})e^{\tilde{x}_{k-1}} + e^{x_k}}{\lambda e^{-x_k + \tilde{x}_{k-1}} + 1} = L_k(x, p; \lambda)[e^{\tilde{x}_{k-1}}].$$

This is equivalent to

$$L_k(x, p; \lambda) \begin{pmatrix} e^{\tilde{x}_{k-1}} \\ 1 \end{pmatrix} \sim \begin{pmatrix} e^{\tilde{x}_k} \\ 1 \end{pmatrix}.$$

The proportionality coefficient is determined by comparing the second components of these two vectors:

$$L_k(x, p; \lambda) \begin{pmatrix} e^{\tilde{x}_{k-1}} \\ 1 \end{pmatrix} = (1 + \lambda e^{\tilde{x}_{k-1} - x_k}) \begin{pmatrix} e^{\tilde{x}_k} \\ 1 \end{pmatrix}.$$

Now the claim in the periodic case follows immediately, with the corresponding eigenvector of $T_N(x, p; \lambda)$ being $(e^{\tilde{x}_N}, 1)^T$. In the open-end case, we have to use additionally the following relations:

$$L_1(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda e^{-x_1} \begin{pmatrix} e^{\tilde{x}_1} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\tilde{x}_N} \\ 1 \end{pmatrix} = e^{\tilde{x}_N}.$$

As a consequence,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} T_N(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda e^{\tilde{x}_N - x_1} \prod_{k=1}^{N-1} (1 + \lambda e^{\tilde{x}_k - x_{k+1}}). \quad \blacksquare$$

7. BÄCKLUND TRANSFORMATIONS FOR SYMMETRIC RATIONAL ADDITIVE TODA-TYPE SYSTEM

The next example constitute Bäcklund transformations for the symmetric rational additive Toda-type system (10) which are given by equations of the type (1):

$$F_\lambda : \begin{cases} p_k = \frac{\lambda}{\tilde{x}_k - x_k} + \frac{\lambda}{x_k - \tilde{x}_{k-1}}, \\ \tilde{p}_k = \frac{\lambda}{\tilde{x}_k - x_k} + \frac{\lambda}{x_{k+1} - \tilde{x}_k}. \end{cases} \quad (34)$$

The corresponding Lagrangian is given by

$$\Lambda(x, \tilde{x}; \lambda) = \lambda \sum_{k=1}^N \log |\tilde{x}_k - x_k| - \lambda \sum_{k=1}^N \log |x_{k+1} - \tilde{x}_k|. \quad (35)$$

The standard single-time Euler-Lagrange equations are (11) with $h = \lambda$. In the open-end case, all terms with $x_1 - \tilde{x}_0$ and with $x_{N+1} - \tilde{x}_N$ should be omitted both from the equations of motion (34) and from the Lagrangian (35). As usual, in the open-end case the first equations in (34) are uniquely solved for $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_N$ (in this order) in terms of (x, p) , while in the periodic case all \tilde{x}_k can be expressed as infinite periodic continued fractions and are, therefore, double-valued functions of (x, p) .

To establish commutativity of the maps F_λ, F_μ , we consider the system of corner equations:

$$\frac{\lambda}{\tilde{x}_k - x_k} + \frac{\lambda}{x_k - \tilde{x}_{k-1}} = \frac{\mu}{\hat{x}_k - x_k} + \frac{\mu}{x_k - \hat{x}_{k-1}}, \quad (E)$$

$$\frac{\lambda}{\tilde{x}_k - x_k} + \frac{\lambda}{x_{k+1} - \tilde{x}_k} = \frac{\mu}{\hat{x}_k - \tilde{x}_k} + \frac{\mu}{\tilde{x}_k - \hat{x}_{k-1}}, \quad (E_1)$$

$$\frac{\mu}{\hat{x}_k - x_k} + \frac{\mu}{x_{k+1} - \hat{x}_k} = \frac{\lambda}{\hat{x}_k - \hat{x}_k} + \frac{\lambda}{\hat{x}_k - \hat{x}_{k-1}}, \quad (E_2)$$

$$\frac{\lambda}{\hat{x}_k - \hat{x}_k} + \frac{\lambda}{\hat{x}_{k+1} - \hat{x}_k} = \frac{\mu}{\hat{x}_k - \tilde{x}_k} + \frac{\mu}{\tilde{x}_{k+1} - \hat{x}_k}. \quad (E_{12})$$

Theorem 10. *Suppose that the fields x, \tilde{x}, \hat{x} satisfy corner equations (E). Define the fields $\hat{\tilde{x}}$ by any of the following two superposition formulas, which are equivalent by virtue of (E):*

$$\mu(\hat{\tilde{x}}_k - \hat{x}_k)(x_{k+1} - \tilde{x}_k) = \lambda(\hat{\tilde{x}}_k - \tilde{x}_k)(x_{k+1} - \hat{x}_k), \quad (S1)$$

$$\mu(\tilde{x}_{k+1} - x_{k+1})(\hat{x}_{k+1} - \hat{\tilde{x}}_k) = \lambda(\hat{x}_{k+1} - x_{k+1})(\tilde{x}_{k+1} - \hat{\tilde{x}}_k). \quad (S2)$$

Then the corner equations (E₁)–(E₁₂) are satisfied, as well.

Proof. Observe that equations (S1) and (S2) are quad-equations with respect to

$$(x_{k+1}, \tilde{x}_k, \hat{x}_k, \hat{\tilde{x}}_k), \quad \text{resp.} \quad (x_{k+1}, \tilde{x}_{k+1}, \hat{x}_{k+1}, \hat{\tilde{x}}_k),$$

namely the cross-ratio equations (Q1_{δ=0} in the notation of ABS list [2]).

The three-leg forms of these equations, centered at x_{k+1} , are:

$$\frac{\lambda - \mu}{\hat{\tilde{x}}_k - x_{k+1}} = \frac{\lambda}{\tilde{x}_k - x_{k+1}} - \frac{\mu}{\hat{x}_k - x_{k+1}},$$

and

$$\frac{\lambda - \mu}{\widehat{\widehat{x}}_k - x_{k+1}} = \frac{\lambda}{\widetilde{x}_{k+1} - x_{k+1}} - \frac{\mu}{\widehat{x}_{k+1} - x_{k+1}},$$

respectively. Their difference coincides with (E) .

The three-leg forms of equations $(S1)$ and $(S2)$, centered at \widetilde{x}_k , resp. at \widetilde{x}_{k+1} , are:

$$\frac{\lambda - \mu}{\widehat{x}_k - \widetilde{x}_k} = \frac{\lambda}{x_{k+1} - \widetilde{x}_k} - \frac{\mu}{\widehat{\widehat{x}}_k - \widetilde{x}_k},$$

and

$$\frac{\lambda - \mu}{\widehat{x}_{k+1} - \widetilde{x}_{k+1}} = \frac{\lambda}{x_{k+1} - \widetilde{x}_{k+1}} - \frac{\mu}{\widehat{\widehat{x}}_k - \widetilde{x}_{k+1}},$$

respectively. The difference of these two equations (the second one with k replaced by $k - 1$) coincides with (E_1) .

The three-leg forms of equations $(S1)$ and $(S2)$, centered at $\widehat{\widehat{x}}_k$, are:

$$\frac{\lambda - \mu}{x_{k+1} - \widehat{\widehat{x}}_k} = \frac{\lambda}{\widehat{x}_k - \widehat{\widehat{x}}_k} - \frac{\mu}{\widetilde{x}_k - \widehat{\widehat{x}}_k},$$

and

$$\frac{\lambda - \mu}{x_{k+1} - \widehat{\widehat{x}}_k} = \frac{\lambda}{\widehat{x}_{k+1} - \widehat{\widehat{x}}_k} - \frac{\mu}{\widetilde{x}_{k+1} - \widehat{\widehat{x}}_k},$$

respectively. The difference of these two equations coincides with (E_{12}) . ■

Theorem 11. *The discrete multi-time Lagrangian 1-form is closed on any solution of the corner equations $(E)-(E_{12})$.*

Proof. With the help of the spectrality criterium, we see that the claim of the theorem is equivalent to

$$P(x, \widetilde{x}) = P(\widehat{x}, \widehat{\widehat{x}}), \quad (36)$$

where in the periodic case

$$P(x, \widetilde{x}) = \frac{\prod_{k=1}^N (x_{k+1} - \widetilde{x}_k)}{\prod_{k=1}^N (\widetilde{x}_k - x_k)},$$

while in the open-end case the product in the numerator of the latter formula is over $1 \leq k \leq N - 1$ only. To prove this relation, we re-write the cross-ratio equations $(S1)$, $(S2)$ in the following equivalent forms:

$$(\lambda - \mu)(\widehat{\widehat{x}}_k - \widehat{x}_k)(x_{k+1} - \widetilde{x}_k) = \lambda(\widetilde{x}_k - \widehat{x}_k)(x_{k+1} - \widehat{\widehat{x}}_k),$$

resp.

$$(\lambda - \mu)(\widehat{x}_{k+1} - \widehat{\widehat{x}}_k)(\widetilde{x}_{k+1} - x_{k+1}) = \lambda(\widetilde{x}_{k+1} - \widehat{x}_{k+1})(x_{k+1} - \widehat{\widehat{x}}_k).$$

As a consequence, we arrive at the following superposition formula:

$$\frac{\widehat{x}_{k+1} - \widehat{\widehat{x}}_k}{\widehat{\widehat{x}}_k - \widehat{x}_k} = \frac{x_{k+1} - \widetilde{x}_k}{\widetilde{x}_{k+1} - x_{k+1}} \cdot \frac{\widetilde{x}_{k+1} - \widehat{x}_{k+1}}{\widetilde{x}_k - \widehat{x}_k}. \quad (37)$$

In the periodic case this formula yields (36). In the open-end case equation (37) holds true for $1 \leq k \leq N-1$, and has to be supplemented with

$$\frac{\lambda}{\tilde{x}_N - \hat{x}_N} = \frac{\lambda - \mu}{\tilde{x}_N - \hat{x}_N}, \quad \frac{\lambda}{\tilde{x}_1 - x_1} = \frac{\lambda - \mu}{\tilde{x}_1 - \hat{x}_1}. \quad (38)$$

Product of equations (37) with $1 \leq k \leq N-1$ with equations (38) yields the claim of the theorem in the open-end case, as well. ■

Theorem 12. *Set*

$$L_k(x, p; \lambda) = I + \lambda^{-1} \begin{pmatrix} p_k x_k & -p_k x_k^2 \\ p_k & -p_k x_k \end{pmatrix},$$

and

$$T_N(x, p; \lambda) = L_N(x, p; \lambda) \cdots L_2(x, p; \lambda) L_1(x, p; \lambda).$$

Then in the periodic case the quantity

$$\frac{\prod_{k=1}^N (\tilde{x}_k - x_{k+1})}{\prod_{k=1}^N (\tilde{x}_k - x_k)}$$

is an eigenvalue of $T_N(x, p; \lambda)$, while in the open-end case its counterpart,

$$\frac{\prod_{k=1}^{N-1} (\tilde{x}_k - x_{k+1})}{\prod_{k=1}^N (\tilde{x}_k - x_k)},$$

is equal to the (21)-entry of $T_N(x, p; \lambda)$.

Proof. We can re-write the first equation in (34) as

$$\tilde{x}_k = x_k + \frac{\lambda}{p_k - \frac{\lambda}{x_k - \tilde{x}_{k-1}}} = \frac{p_k x_k (x_k - \tilde{x}_{k-1}) - \lambda \tilde{x}_{k-1}}{p_k (x_k - \tilde{x}_{k-1}) - \lambda} = L_k(x, p; \lambda) [\tilde{x}_{k-1}].$$

This is equivalent to

$$L_k(x, p; \lambda) \begin{pmatrix} \tilde{x}_{k-1} \\ 1 \end{pmatrix} \sim \begin{pmatrix} \tilde{x}_k \\ 1 \end{pmatrix}.$$

The proportionality coefficient is determined by comparing the second components of these two vectors and is equal to

$$1 - \lambda^{-1} p_k (x_k - \tilde{x}_{k-1}) = 1 - (x_k - \tilde{x}_{k-1}) \left(\frac{1}{\tilde{x}_k - x_k} + \frac{1}{x_k - \tilde{x}_{k-1}} \right) = \frac{\tilde{x}_{k-1} - x_k}{\tilde{x}_k - x_k}.$$

Thus,

$$L_k(x, p; \lambda) \begin{pmatrix} \tilde{x}_{k-1} \\ 1 \end{pmatrix} = \frac{\tilde{x}_{k-1} - x_k}{\tilde{x}_k - x_k} \begin{pmatrix} \tilde{x}_k \\ 1 \end{pmatrix}. \quad (39)$$

Now the claim in the periodic case follows immediately, with the corresponding eigenvector of $T_N(x, p; \lambda)$ being $(\tilde{x}_N, 1)^T$. In the open-end case, equation (39) holds true for $2 \leq k \leq N$, and has to be supplemented by the following two relations:

$$L_1(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\tilde{x}_1 - x_1} \begin{pmatrix} \tilde{x}_1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_N \\ 1 \end{pmatrix} = 1.$$

As a consequence,

$$(0 \ 1) T_N(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\prod_{k=1}^{N-1} (\tilde{x}_k - x_{k+1})}{\prod_{k=1}^N (\tilde{x}_k - x_k)}. \quad \blacksquare$$

8. BÄCKLUND TRANSFORMATIONS FOR SYMMETRIC RATIONAL MULTIPLICATIVE TODA-TYPE SYSTEM

The next example constitute Bäcklund transformations for the symmetric rational multiplicative Toda-type system (12) which are given by equations of the type (1):

$$F_\lambda : \begin{cases} e^{2p_k} = \frac{\tilde{x}_k - x_k + \lambda}{\tilde{x}_k - x_k - \lambda} \cdot \frac{x_k - \tilde{x}_{k-1} + \lambda}{x_k - \tilde{x}_{k-1} - \lambda}, \\ e^{2\tilde{p}_k} = \frac{\tilde{x}_k - x_k + \lambda}{\tilde{x}_k - x_k - \lambda} \cdot \frac{x_{k+1} - \tilde{x}_k + \lambda}{x_{k+1} - \tilde{x}_k - \lambda}. \end{cases} \quad (40)$$

The corresponding Lagrangian is given by

$$\Lambda(x, \tilde{x}; \lambda) = \sum_{k=1}^N \psi(\tilde{x}_k - x_k; \lambda) - \sum_{k=1}^N \psi(x_{k+1} - \tilde{x}_k; \lambda), \quad (41)$$

where

$$\psi(\xi; \lambda) = \frac{1}{2} \int_{\xi-\lambda}^{\xi+\lambda} \log \eta d\eta.$$

The standard single-time Euler-Lagrange equations are (13) with $h = \lambda$. In the open-end case, all terms with $x_1 - \tilde{x}_0$ and with $x_{N+1} - \tilde{x}_N$ should be omitted both from the equations of motion (40) and from the Lagrangian (41).

To establish commutativity of the maps F_λ, F_μ , we consider the system of corner equations:

$$\frac{\tilde{x}_k - x_k + \lambda}{\tilde{x}_k - x_k - \lambda} \cdot \frac{x_k - \tilde{x}_{k-1} + \lambda}{x_k - \tilde{x}_{k-1} - \lambda} = \frac{\hat{x}_k - x_k + \mu}{\hat{x}_k - x_k - \mu} \cdot \frac{x_k - \hat{x}_{k-1} + \mu}{x_k - \hat{x}_{k-1} - \mu}, \quad (E)$$

$$\frac{\tilde{x}_k - x_k + \lambda}{\tilde{x}_k - x_k - \lambda} \cdot \frac{x_{k+1} - \tilde{x}_k + \lambda}{x_{k+1} - \tilde{x}_k - \lambda} = \frac{\hat{\tilde{x}}_k - \tilde{x}_k + \mu}{\hat{\tilde{x}}_k - \tilde{x}_k - \mu} \cdot \frac{\tilde{x}_k - \hat{\tilde{x}}_{k-1} + \mu}{\tilde{x}_k - \hat{\tilde{x}}_{k-1} - \mu}, \quad (E_1)$$

$$\frac{\hat{x}_k - x_k + \mu}{\hat{x}_k - x_k - \mu} \cdot \frac{x_{k+1} - \hat{x}_k + \mu}{x_{k+1} - \hat{x}_k - \mu} = \frac{\hat{\hat{x}}_k - \hat{x}_k + \lambda}{\hat{\hat{x}}_k - \hat{x}_k - \lambda} \cdot \frac{\hat{x}_k - \hat{\hat{x}}_{k-1} + \lambda}{\hat{x}_k - \hat{\hat{x}}_{k-1} - \lambda}, \quad (E_2)$$

$$\frac{\hat{\tilde{x}}_k - \hat{\tilde{x}}_k + \lambda}{\hat{\tilde{x}}_k - \hat{\tilde{x}}_k - \lambda} \cdot \frac{\hat{\tilde{x}}_{k+1} - \hat{\tilde{x}}_k + \lambda}{\hat{\tilde{x}}_{k+1} - \hat{\tilde{x}}_k - \lambda} = \frac{\hat{\tilde{x}}_k - \tilde{x}_k + \mu}{\hat{\tilde{x}}_k - \tilde{x}_k - \mu} \cdot \frac{\tilde{x}_{k+1} - \hat{\tilde{x}}_k + \mu}{\tilde{x}_{k+1} - \hat{\tilde{x}}_k - \mu}. \quad (E_{12})$$

Theorem 13. Suppose that the fields x, \tilde{x}, \hat{x} satisfy corner equations (E). Define the fields $\hat{\tilde{x}}$ by any of the following two superposition formulas, which are equivalent by virtue of (E):

$$\mu(\hat{\tilde{x}}_k - \hat{x}_k)(x_{k+1} - \tilde{x}_k) - \lambda(\hat{\tilde{x}}_k - \tilde{x}_k)(x_{k+1} - \hat{x}_k) + \lambda\mu(\lambda - \mu) = 0, \quad (S1)$$

$$\mu(\tilde{x}_{k+1} - x_{k+1})(\hat{\tilde{x}}_{k+1} - \hat{x}_k) - \lambda(\hat{\tilde{x}}_{k+1} - x_{k+1})(\tilde{x}_{k+1} - \hat{\tilde{x}}_k) + \lambda\mu(\lambda - \mu) = 0. \quad (S2)$$

Then the corner equations (E₁)–(E₁₂) are satisfied, as well.

Proof. Observe that equations (S1) and (S2) are quad-equations with respect to

$$(x_{k+1}, \tilde{x}_k, \hat{x}_k, \hat{\tilde{x}}_k), \quad \text{resp.} \quad (x_{k+1}, \tilde{x}_{k+1}, \hat{x}_{k+1}, \hat{\tilde{x}}_k),$$

namely of the type Q1_{δ=1} from the ABS list [2].

The three-leg forms of these equations, centered at x_{k+1} , are:

$$\frac{x_{k+1} - \tilde{x}_k + \lambda}{x_{k+1} - \tilde{x}_k - \lambda} = \frac{x_{k+1} - \hat{x}_k + \mu}{x_{k+1} - \hat{x}_k - \mu} \cdot \frac{\hat{\tilde{x}}_k - x_{k+1} - \lambda + \mu}{\hat{\tilde{x}}_k - x_{k+1} + \lambda - \mu},$$

and

$$\frac{\tilde{x}_{k+1} - x_{k+1} + \lambda}{\tilde{x}_{k+1} - x_{k+1} - \lambda} = \frac{\hat{x}_{k+1} - x_{k+1} + \mu}{\hat{x}_{k+1} - x_{k+1} - \mu} \cdot \frac{\hat{\tilde{x}}_k - x_{k+1} + \lambda - \mu}{\hat{\tilde{x}}_k - x_{k+1} - \lambda + \mu}$$

respectively. Their product coincides with (E) with k replaced by $k+1$.

The three-leg forms of equations (S1) and (S2), centered at \tilde{x}_k , resp. at \tilde{x}_{k+1} , are:

$$\frac{x_{k+1} - \tilde{x}_k + \lambda}{x_{k+1} - \tilde{x}_k - \lambda} = \frac{\hat{\tilde{x}}_k - \tilde{x}_k + \mu}{\hat{\tilde{x}}_k - \tilde{x}_k - \mu} \cdot \frac{\hat{x}_k - \tilde{x}_k + \lambda - \mu}{\hat{x}_k - \tilde{x}_k - \lambda + \mu},$$

and

$$\frac{\tilde{x}_{k+1} - x_{k+1} + \lambda}{\tilde{x}_{k+1} - x_{k+1} - \lambda} = \frac{\tilde{x}_{k+1} - \hat{\tilde{x}}_k + \mu}{\tilde{x}_{k+1} - \hat{\tilde{x}}_k - \mu} \cdot \frac{\hat{x}_{k+1} - \tilde{x}_{k+1} - \lambda + \mu}{\hat{x}_{k+1} - \tilde{x}_{k+1} + \lambda - \mu},$$

respectively. The product of these two equations (the second one with k replaced by $k-1$) coincides with (E₁).

The three-leg forms of equations (S1) and (S2), centered at $\hat{\tilde{x}}_k$, are:

$$\frac{\hat{\tilde{x}}_k - \hat{x}_k + \lambda}{\hat{\tilde{x}}_k - \hat{x}_k - \lambda} = \frac{\hat{\tilde{x}}_k - \tilde{x}_k + \mu}{\hat{\tilde{x}}_k - \tilde{x}_k - \mu} \cdot \frac{\hat{\tilde{x}}_k - x_{k+1} + \lambda - \mu}{\hat{\tilde{x}}_k - x_{k+1} - \lambda + \mu},$$

and

$$\frac{\hat{x}_{k+1} - \hat{\tilde{x}}_k + \lambda}{\hat{x}_{k+1} - \hat{\tilde{x}}_k - \lambda} = \frac{\tilde{x}_{k+1} - \hat{\tilde{x}}_k + \mu}{\tilde{x}_{k+1} - \hat{\tilde{x}}_k - \mu} \cdot \frac{\hat{\tilde{x}}_k - x_{k+1} - \lambda + \mu}{\hat{\tilde{x}}_k - x_{k+1} + \lambda - \mu},$$

respectively. The product of these two equations coincides with (E₁₂). ■

Theorem 14. *The discrete multi-time Lagrangian 1-form is closed on any solution of the corner equations (E)–(E₁₂).*

Proof. Spectrality criterium requires to prove that the following quantity is an integral of motion for F_μ :

$$P(x, \tilde{x}) = \exp(-2\partial\Lambda(x, \tilde{x}; \lambda)/\partial\lambda) = \frac{\prod_{k=1}^N ((x_{k+1} - \tilde{x}_k)^2 - \lambda^2)}{\prod_{k=1}^N ((\tilde{x}_k - x_k)^2 - \lambda^2)}$$

(in the periodic case; in the open-end case the product in the numerator of the latter formula is over $1 \leq k \leq N-1$ only). Taking into account the obvious integral of motion

$$\prod_{k=1}^N e^{2p_k} = \prod_{k=1}^N \frac{\tilde{x}_k - x_k + \lambda}{\tilde{x}_k - x_k - \lambda} \cdot \prod_{k=1}^N \frac{x_{k+1} - \tilde{x}_k + \lambda}{x_{k+1} - \tilde{x}_k - \lambda},$$

we see that we have to prove the property

$$P(x, \tilde{x}) = P(\hat{x}, \hat{\tilde{x}}) \tag{42}$$

for the quantity for either of the following two quantities:

$$P_1(x, \tilde{x}) = \frac{\prod_{k=1}^N (x_{k+1} - \tilde{x}_k + \lambda)}{\prod_{k=1}^N (\tilde{x}_k - x_k - \lambda)}, \quad (43)$$

$$P_2(x, \tilde{x}) = \frac{\prod_{k=1}^N (x_{k+1} - \tilde{x}_k - \lambda)}{\prod_{k=1}^N (\tilde{x}_k - x_k + \lambda)}. \quad (44)$$

To prove this, we re-write superposition formulas (S1), (S2) in the following equivalent forms:

$$(\lambda - \mu)(\widehat{x}_k - \widehat{x}_k \mp \lambda)(x_{k+1} - \tilde{x}_k \pm \lambda) = \lambda(\tilde{x}_k - \widehat{x}_k \mp \lambda \pm \mu)(x_{k+1} - \widehat{x}_k \pm \lambda \mp \mu),$$

and

$$(\lambda - \mu)(\widehat{x}_{k+1} - \widehat{x}_k \pm \lambda)(\tilde{x}_{k+1} - x_{k+1} \mp \lambda) = \lambda(\tilde{x}_{k+1} - \widehat{x}_{k+1} \mp \lambda \pm \mu)(x_{k+1} - \widehat{x}_k \pm \lambda \mp \mu).$$

As a consequence, we arrive at the following superposition formula:

$$\frac{\widehat{x}_{k+1} - \widehat{x}_k \pm \lambda}{\widehat{x}_k - \widehat{x}_k \mp \lambda} = \frac{x_{k+1} - \tilde{x}_k \pm \lambda}{\tilde{x}_{k+1} - x_{k+1} \mp \lambda} \cdot \frac{\tilde{x}_{k+1} - \widehat{x}_{k+1} \mp \lambda \pm \mu}{\tilde{x}_k - \widehat{x}_k \mp \lambda \pm \mu}. \quad (45)$$

In the periodic case, the latter formula yields (42) for both quantities (43), (44). In the open-end case equation (45) holds true for $1 \leq k \leq N-1$, and has to be supplemented with the following two relations:

$$\frac{\lambda}{\widehat{x}_N - \widehat{x}_N \mp \lambda} = \frac{\lambda - \mu}{\tilde{x}_N - \widehat{x}_N \mp \lambda \pm \mu}, \quad \frac{\lambda}{\tilde{x}_1 - x_1 \mp \lambda} = \frac{\lambda - \mu}{\tilde{x}_1 - \widehat{x}_1 \mp \lambda \pm \mu},$$

which are equivalent to equation (E_{12}) for $k = N$, resp. to equation (E) for $k = 1$. ■

Theorem 15. *Set*

$$L_k(x, p; \lambda) = \begin{pmatrix} \lambda(e^{2p_k} + 1) + x_k(e^{2p_k} - 1) & (\lambda^2 - x_k^2)(e^{2p_k} - 1) \\ e^{2p_k} - 1 & \lambda(e^{2p_k} + 1) - x_k(e^{2p_k} - 1) \end{pmatrix},$$

and

$$T_N(x, p; \lambda) = L_N(x, p; \lambda) \cdots L_2(x, p; \lambda) L_1(x, p; \lambda).$$

Then in the periodic case quantity

$$(2\lambda)^N \frac{\prod_{k=1}^N (\tilde{x}_k - x_{k+1} - \lambda)}{\prod_{k=1}^N (\tilde{x}_k - x_k - \lambda)},$$

is an eigenvalue of $T_N(x, p; \lambda)$, while in the open-end case its counterpart,

$$(2\lambda)^N \frac{\prod_{k=1}^{N-1} (\tilde{x}_k - x_{k+1} - \lambda)}{\prod_{k=1}^N (\tilde{x}_k - x_k - \lambda)},$$

is equal to the (21) -entry of $T_N(x, p; \lambda)$.

Proof. We can re-write the first equation in (40) as

$$\begin{aligned} \tilde{x}_k &= \frac{e^{2p_k}(\lambda + x_k)(x_k - \tilde{x}_{k-1} - \lambda) + (\lambda - x_k)(x_k - \tilde{x}_{k-1} + \lambda)}{e^{2p_k}(x_k - \tilde{x}_{k-1} - \lambda) - (x_k - \tilde{x}_{k-1} + \lambda)} \\ &= \frac{(\lambda(e^{2p_k} + 1) + x_k(e^{2p_k} - 1))\tilde{x}_{k-1} + (\lambda^2 - x_k^2)(e^{2p_k} - 1)}{(e^{2p_k} - 1)\tilde{x}_{k-1} + \lambda(e^{2p_k} + 1) - x_k(e^{2p_k} - 1)} \\ &= L_k(x, p; \lambda)[\tilde{x}_{k-1}]. \end{aligned}$$

This is equivalent to

$$L_k(x, p; \lambda) \begin{pmatrix} \tilde{x}_{k-1} \\ 1 \end{pmatrix} \sim \begin{pmatrix} \tilde{x}_k \\ 1 \end{pmatrix}.$$

The proportionality coefficient is determined by comparing the second components of these two vectors and is equal to

$$\begin{aligned} & (e^{2p_k} - 1)(\tilde{x}_{k-1} - x_k) + \lambda(e^{2p_k} + 1) \\ &= (x_k - \tilde{x}_{k-1} + \lambda) - e^{2p_k}(x_k - \tilde{x}_{k-1} - \lambda) = -2\lambda \frac{x_k - \tilde{x}_{k-1} + \lambda}{\tilde{x}_k - x_k - \lambda}. \end{aligned}$$

Thus,

$$L_k(x, p; \lambda) \begin{pmatrix} \tilde{x}_{k-1} \\ 1 \end{pmatrix} = 2\lambda \frac{\tilde{x}_{k-1} - x_k - \lambda}{\tilde{x}_k - x_k - \lambda} \begin{pmatrix} \tilde{x}_k \\ 1 \end{pmatrix}. \quad (46)$$

Now the claim in the periodic case follows immediately, with the corresponding eigenvector of $T_N(x, p; \lambda)$ being $(\tilde{x}_N, 1)^T$. In the open-end case, equation (46) holds true for $2 \leq k \leq N$, and has to be supplemented by the following two relations:

$$L_1(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2\lambda}{\tilde{x}_1 - x_1 - \lambda} \begin{pmatrix} \tilde{x}_1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_N \\ 1 \end{pmatrix} = 1.$$

As a consequence,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} T_N(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (2\lambda)^N \frac{\prod_{k=1}^{N-1} (\tilde{x}_k - x_{k+1} - \lambda)}{\prod_{k=1}^N (\tilde{x}_k - x_k - \lambda)}. \quad \blacksquare$$

9. BÄCKLUND TRANSFORMATIONS FOR SYMMETRIC HYPERBOLIC MULTIPLICATIVE TODA-TYPE SYSTEM

Our last example constitutes Bäcklund transformations for the symmetric hyperbolic multiplicative Toda-type system (14) which are given by equations of the type (1):

$$F_\lambda : \begin{cases} e^{2p_k} = \frac{\sinh(\tilde{x}_k - x_k + \lambda)}{\sinh(\tilde{x}_k - x_k - \lambda)} \cdot \frac{\sinh(x_k - \tilde{x}_{k-1} + \lambda)}{\sinh(x_k - \tilde{x}_{k-1} - \lambda)}, \\ e^{2\tilde{p}_k} = \frac{\sinh(\tilde{x}_k - x_k + \lambda)}{\sinh(\tilde{x}_k - x_k - \lambda)} \cdot \frac{\sinh(x_{k+1} - \tilde{x}_k + \lambda)}{\sinh(x_{k+1} - \tilde{x}_k - \lambda)}. \end{cases} \quad (47)$$

The corresponding Lagrangian is given by

$$\Lambda(x, \tilde{x}; \lambda) = \sum_{k=1}^N \psi(\tilde{x}_k - x_k; \lambda) - \sum_{k=1}^N \psi(x_{k+1} - \tilde{x}_k; \lambda), \quad (48)$$

where

$$\psi(\xi; \lambda) = \frac{1}{2} \int_{\xi-\lambda}^{\xi+\lambda} \log \sinh(\eta) d\eta.$$

The standard single-time Euler-Lagrange equations are (15) with $h = \lambda$. In the open-end case, all terms with $x_1 - \tilde{x}_0$ and with $x_{N+1} - \tilde{x}_N$ should be omitted both from the equations of motion (47) and from the Lagrangian (48).

To establish commutativity of the maps F_λ , F_μ , we consider the system of corner equations:

$$\frac{\sinh(\tilde{x}_k - x_k + \lambda)}{\sinh(\tilde{x}_k - x_k - \lambda)} \cdot \frac{\sinh(x_k - \tilde{x}_{k-1} + \lambda)}{\sinh(x_k - \tilde{x}_{k-1} - \lambda)} = \frac{\sinh(\hat{x}_k - x_k + \mu)}{\sinh(\hat{x}_k - x_k - \mu)} \cdot \frac{\sinh(x_k - \hat{x}_{k-1} + \mu)}{\sinh(x_k - \hat{x}_{k-1} - \mu)}, \quad (E)$$

$$\frac{\sinh(\tilde{x}_k - x_k + \lambda)}{\sinh(\tilde{x}_k - x_k - \lambda)} \cdot \frac{\sinh(x_{k+1} - \tilde{x}_k + \lambda)}{\sinh(x_{k+1} - \tilde{x}_k - \lambda)} = \frac{\sinh(\hat{\tilde{x}}_k - \tilde{x}_k + \mu)}{\sinh(\hat{\tilde{x}}_k - \tilde{x}_k - \mu)} \cdot \frac{\sinh(\tilde{x}_k - \hat{\tilde{x}}_{k-1} + \mu)}{\sinh(\tilde{x}_k - \hat{\tilde{x}}_{k-1} - \mu)}, \quad (E_1)$$

$$\frac{\sinh(\hat{x}_k - x_k + \mu)}{\sinh(\hat{x}_k - x_k - \mu)} \cdot \frac{\sinh(x_{k+1} - \hat{x}_k + \mu)}{\sinh(x_{k+1} - \hat{x}_k - \mu)} = \frac{\sinh(\hat{\tilde{x}}_k - \hat{x}_k + \lambda)}{\sinh(\hat{\tilde{x}}_k - \hat{x}_k - \lambda)} \cdot \frac{\sinh(\hat{x}_k - \hat{\tilde{x}}_{k-1} + \lambda)}{\sinh(\hat{x}_k - \hat{\tilde{x}}_{k-1} - \lambda)}, \quad (E_2)$$

$$\frac{\sinh(\hat{\tilde{x}}_k - \hat{x}_k + \lambda)}{\sinh(\hat{\tilde{x}}_k - \hat{x}_k - \lambda)} \cdot \frac{\sinh(\hat{x}_{k+1} - \hat{\tilde{x}}_k + \lambda)}{\sinh(\hat{x}_{k+1} - \hat{\tilde{x}}_k - \lambda)} = \frac{\sinh(\hat{\tilde{x}}_k - \tilde{x}_k + \mu)}{\sinh(\hat{\tilde{x}}_k - \tilde{x}_k - \mu)} \cdot \frac{\sinh(\tilde{x}_{k+1} - \hat{\tilde{x}}_k + \mu)}{\sinh(\tilde{x}_{k+1} - \hat{\tilde{x}}_k - \mu)}. \quad (E_{12})$$

Theorem 16. Suppose that the fields x , \tilde{x} , \hat{x} satisfy corner equations (E). Define the fields $\hat{\tilde{x}}$ by any of the following two superposition formulas, which are equivalent by virtue of (E):

$$(e^{4\lambda} - e^{4\mu})(e^{2\hat{\tilde{x}}_k} e^{2\tilde{x}_k} + e^{2x_{k+1}} e^{2\hat{\tilde{x}}_k}) + e^{2\mu}(1 - e^{4\lambda})(e^{2\hat{\tilde{x}}_k} e^{2\tilde{x}_k} + e^{2x_{k+1}} e^{2\tilde{x}_k}) + e^{2\lambda}(e^{4\mu} - 1)(e^{2\tilde{x}_k} e^{2\hat{\tilde{x}}_k} + e^{2x_{k+1}} e^{2\tilde{x}_k}) = 0, \quad (S1)$$

$$(e^{4\lambda} - e^{4\mu})(e^{2\hat{\tilde{x}}_{k+1}} e^{2\tilde{x}_{k+1}} + e^{2x_{k+1}} e^{2\hat{\tilde{x}}_{k+1}}) + e^{2\mu}(1 - e^{4\lambda})(e^{2\hat{\tilde{x}}_{k+1}} e^{2\tilde{x}_{k+1}} + e^{2x_{k+1}} e^{2\tilde{x}_{k+1}}) + e^{2\lambda}(e^{4\mu} - 1)(e^{2\tilde{x}_{k+1}} e^{2\hat{\tilde{x}}_{k+1}} + e^{2x_{k+1}} e^{2\tilde{x}_{k+1}}) = 0. \quad (S2)$$

Then corner equations (E₁)–(E₁₂) are satisfied, as well.

Proof. Observe that equations (S1) and (S2) are quad-equations with respect to

$$(e^{2x_{k+1}}, e^{2\tilde{x}_k}, e^{2\hat{\tilde{x}}_k}, e^{2\tilde{x}_k}), \quad \text{resp.} \quad (e^{2x_{k+1}}, e^{2\tilde{x}_{k+1}}, e^{2\hat{\tilde{x}}_{k+1}}, e^{2\tilde{x}_{k+1}}),$$

namely of the type Q3_{δ=0} from the ABS list [2].

The three-leg forms of these equations, centered at x_{k+1} , are:

$$\frac{\sinh(x_{k+1} - \tilde{x}_k + \lambda)}{\sinh(x_{k+1} - \tilde{x}_k - \lambda)} = \frac{\sinh(x_{k+1} - \hat{x}_k + \mu)}{\sinh(x_{k+1} - \hat{x}_k - \mu)} \cdot \frac{\sinh(\hat{\tilde{x}}_k - x_{k+1} - \lambda + \mu)}{\sinh(\hat{\tilde{x}}_k - x_{k+1} + \lambda - \mu)},$$

and

$$\frac{\sinh(\tilde{x}_{k+1} - x_{k+1} + \lambda)}{\sinh(\tilde{x}_{k+1} - x_{k+1} - \lambda)} = \frac{\sinh(\hat{x}_{k+1} - x_{k+1} + \mu)}{\sinh(\hat{x}_{k+1} - x_{k+1} - \mu)} \cdot \frac{\sinh(\hat{\tilde{x}}_k - x_{k+1} + \lambda - \mu)}{\sinh(\hat{\tilde{x}}_k - x_{k+1} - \lambda + \mu)},$$

respectively. Their product coincides with (E) with k replaced by $k+1$.

The three-leg forms of equations (S1) and (S2), centered at \tilde{x}_k , resp. at \tilde{x}_{k+1} , are:

$$\frac{\sinh(x_{k+1} - \tilde{x}_k + \lambda)}{\sinh(x_{k+1} - \tilde{x}_k - \lambda)} = \frac{\sinh(\hat{\tilde{x}}_k - \tilde{x}_k + \mu)}{\sinh(\hat{\tilde{x}}_k - \tilde{x}_k - \mu)} \cdot \frac{\sinh(\hat{x}_k - \tilde{x}_k + \lambda - \mu)}{\sinh(\hat{x}_k - \tilde{x}_k - \lambda + \mu)},$$

and

$$\frac{\sinh(\tilde{x}_{k+1} - x_{k+1} + \lambda)}{\sinh(\tilde{x}_{k+1} - x_{k+1} - \lambda)} = \frac{\sinh(\tilde{x}_{k+1} - \hat{\tilde{x}}_k + \mu)}{\sinh(\tilde{x}_{k+1} - \hat{\tilde{x}}_k - \mu)} \cdot \frac{\sinh(\hat{x}_{k+1} - \tilde{x}_{k+1} - \lambda + \mu)}{\sinh(\hat{x}_{k+1} - \tilde{x}_{k+1} + \lambda - \mu)},$$

respectively. The product of these two equations (the second one with k replaced by $k-1$) coincides with (E₁).

The three-leg forms of equations (S1) and (S2), centered at $\hat{\tilde{x}}_k$, are:

$$\frac{\sinh(\hat{\tilde{x}}_k - \hat{x}_k + \lambda)}{\sinh(\hat{\tilde{x}}_k - \hat{x}_k - \lambda)} = \frac{\sinh(\hat{\tilde{x}}_k - \tilde{x}_k + \mu)}{\sinh(\hat{\tilde{x}}_k - \tilde{x}_k - \mu)} \cdot \frac{\sinh(\hat{\tilde{x}}_k - x_{k+1} + \lambda - \mu)}{\sinh(\hat{\tilde{x}}_k - x_{k+1} - \lambda + \mu)},$$

and

$$\frac{\sinh(\widehat{x}_{k+1} - \widehat{x}_k + \lambda)}{\sinh(\widehat{x}_{k+1} - \widehat{x}_k - \lambda)} = \frac{\sinh(\widetilde{x}_{k+1} - \widetilde{x}_k + \mu)}{\sinh(\widetilde{x}_{k+1} - \widetilde{x}_k - \mu)} \cdot \frac{\sinh(\widehat{x}_k - x_{k+1} - \lambda + \mu)}{\sinh(\widehat{x}_k - x_{k+1} + \lambda - \mu)},$$

respectively. The product of these two equations coincides with (E_{12}) . ■

Theorem 17. *The discrete multi-time Lagrangian 1-form is closed on any solution of the corner equations $(E)-(E_{12})$.*

Proof. With the help of the spectrality criterium, we see that the claim of the theorem is equivalent to

$$P(x, \widetilde{x}) = P(\widehat{x}, \widehat{\widetilde{x}}) \quad (49)$$

for the quantity

$$P(x, \widetilde{x}) = \frac{\prod_{k=1}^N \sinh(x_{k+1} - \widetilde{x}_k - \lambda) \sinh(x_{k+1} - \widetilde{x}_k + \lambda)}{\prod_{k=1}^N \sinh(\widetilde{x}_k - x_k - \lambda) \sinh(\widetilde{x}_k - x_k + \lambda)},$$

(in the periodic case; in the open-end case the product in the numerator of the latter formula is over $1 \leq k \leq N-1$ only). Taking into account the obvious integral of motion

$$\prod_{k=1}^N e^{2p_k} = \prod_{k=1}^N \frac{\sinh(\widetilde{x}_k - x_k + \lambda)}{\sinh(\widetilde{x}_k - x_k - \lambda)} \cdot \prod_{k=1}^N \frac{\sinh(x_{k+1} - \widetilde{x}_k + \lambda)}{\sinh(x_{k+1} - \widetilde{x}_k - \lambda)},$$

we see that we have to prove the property (49) for either of the following two quantities:

$$P_1(x, \widetilde{x}) = \frac{\prod_{k=1}^N \sinh(x_{k+1} - \widetilde{x}_k + \lambda)}{\prod_{k=1}^N \sinh(\widetilde{x}_k - x_k - \lambda)}, \quad (50)$$

$$P_2(x, \widetilde{x}) = \frac{\prod_{k=1}^N \sinh(x_{k+1} - \widetilde{x}_k - \lambda)}{\prod_{k=1}^N \sinh(\widetilde{x}_k - x_k + \lambda)}. \quad (51)$$

To prove this relation, we re-write superposition formulas (S1), (S2) in the following equivalent forms:

$$\sinh(\widehat{x}_k - \widehat{x}_k \mp \lambda) \sinh(x_{k+1} - \widetilde{x}_k \pm \lambda) = c \sinh(\widetilde{x}_k - \widehat{x}_k \mp \lambda \pm \mu) \sinh(x_{k+1} - \widehat{x}_k \pm \lambda \mp \mu),$$

resp.

$$\sinh(\widehat{x}_{k+1} - \widehat{x}_k \pm \lambda) \sinh(\widetilde{x}_{k+1} - x_{k+1} \mp \lambda) = c \sinh(\widetilde{x}_{k+1} - \widehat{x}_{k+1} \mp \lambda \pm \mu) \sinh(x_{k+1} - \widehat{x}_k \pm \lambda \mp \mu),$$

where $c = (e^{4\lambda} - 1)/(e^{4\lambda} - e^{4\mu})$. As a consequence, we arrive at the following superposition formula:

$$\frac{\sinh(\widehat{x}_{k+1} - \widehat{x}_k \pm \lambda)}{\sinh(\widehat{x}_k - \widehat{x}_k \mp \lambda)} = \frac{\sinh(x_{k+1} - \widetilde{x}_k \pm \lambda)}{\sinh(\widetilde{x}_{k+1} - x_{k+1} \mp \lambda)} \cdot \frac{\sinh(\widetilde{x}_{k+1} - \widehat{x}_{k+1} \mp \lambda \pm \mu)}{\sinh(\widetilde{x}_k - \widehat{x}_k \mp \lambda \pm \mu)}. \quad (52)$$

In the periodic case, the latter formula yields (49) for both quantities (50), (51). In the open-end case, equation (52) holds true for $1 \leq k \leq N-1$, and has to be supplemented with the following two relations:

$$\frac{c}{\sinh(\widehat{x}_N - \widehat{x}_N \mp \lambda)} = \frac{1}{\sinh(\widetilde{x}_N - \widehat{x}_N \mp \lambda \pm \mu)},$$

$$\frac{c}{\sinh(\widetilde{x}_1 - x_1 \mp \lambda)} = \frac{1}{\sinh(\widetilde{x}_1 - \widehat{x}_1 \mp \lambda \pm \mu)},$$

which are equivalent to equation (E_{12}) for $k = N$, resp. to equation (E) for $k = 1$. ■

Theorem 18. *Set*

$$L_k(x, p; \lambda) = \begin{pmatrix} e^{4\lambda} e^{2p_k} - 1 & e^{2\lambda} e^{2x_k} (1 - e^{2p_k}) \\ e^{2\lambda} e^{-2x_k} (e^{2p_k} - 1) & e^{4\lambda} - e^{2p_k} \end{pmatrix},$$

and

$$T_N(x, p; \lambda) = L_N(x, p; \lambda) \cdots L_2(x, p; \lambda) L_1(x, p; \lambda).$$

Then in the periodic case the quantity

$$(1 - e^{4\lambda})^N \frac{\prod_{k=1}^N \sinh(x_{k+1} - \tilde{x}_k + \lambda)}{\prod_{k=1}^N \sinh(\tilde{x}_k - x_k - \lambda)},$$

is an eigenvalue of $T_N(x, p; \lambda)$, while in the open-end case its counterpart,

$$(1 - e^{4\lambda})^N \frac{\prod_{k=1}^{N-1} \sinh(x_{k+1} - \tilde{x}_k + \lambda)}{\prod_{k=1}^N \sinh(\tilde{x}_k - x_k - \lambda)},$$

is equal to the (21) -entry of $T_N(x, p; \lambda)$.

Proof. We can re-write the first equation in (47) as

$$e^{2\tilde{x}_k} = e^{2x_k} \frac{(e^{4\lambda} e^{2p_k} - 1) e^{2\tilde{x}_{k-1}} - e^{2\lambda} e^{2x_k} (1 - e^{2p_k})}{e^{2\lambda} (e^{2p_k} - 1) e^{2\tilde{x}_{k-1}} + e^{2x_k} (e^{4\lambda} - e^{2p_k})} = L_k(x, p; \lambda) [e^{2\tilde{x}_{k-1}}].$$

This is equivalent to

$$L_k(x, p; \lambda) \begin{pmatrix} e^{2\tilde{x}_{k-1}} \\ 1 \end{pmatrix} \sim \begin{pmatrix} e^{2\tilde{x}_k} \\ 1 \end{pmatrix}.$$

The proportionality coefficient is determined by comparing the second components of these two vectors and is equal to

$$e^{2\lambda} (e^{2p_k} - 1) e^{2\tilde{x}_{k-1} - 2x_k} + e^{4\lambda} - e^{2p_k} = (1 - e^{4\lambda}) \frac{\sinh(x_k - \tilde{x}_{k-1} + \lambda)}{\sinh(\tilde{x}_k - x_k - \lambda)}.$$

Thus,

$$L_k(x, p; \lambda) \begin{pmatrix} e^{2\tilde{x}_{k-1}} \\ 1 \end{pmatrix} = (1 - e^{4\lambda}) \frac{\sinh(x_k - \tilde{x}_{k-1} + \lambda)}{\sinh(\tilde{x}_k - x_k - \lambda)} \begin{pmatrix} e^{2\tilde{x}_k} \\ 1 \end{pmatrix}. \quad (53)$$

Now the claim in the periodic case follows immediately, with the corresponding eigenvector of $T_N(x, p; \lambda)$ being $(e^{2\tilde{x}_N}, 1)^T$. In the open-end case, equation (53) holds true for $2 \leq k \leq N$, and has to be supplemented by the following two relations:

$$L_1(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1 - e^{4\lambda}}{\sinh(\tilde{x}_1 - x_1 - \lambda)} \begin{pmatrix} e^{2\tilde{x}_1} \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2\tilde{x}_N} \\ 1 \end{pmatrix} = 1.$$

As a consequence,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} T_N(x, p; \lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 - e^{4\lambda})^N \frac{\prod_{k=1}^{N-1} \sinh(x_{k+1} - \tilde{x}_k + \lambda)}{\prod_{k=1}^N \sinh(\tilde{x}_k - x_k - \lambda)}. \quad \blacksquare$$

10. CONCLUSIONS

In a forthcoming paper, we will present results on the multi-time Lagrangian one-forms for a more general class of Bäcklund transformations, namely for systems of the relativistic Toda type. This will give us an opportunity to present an alternative approach to this theory, namely an approach from the point of view of two-dimensional integrable systems. Indeed, it is well known since [1] that discrete time relativistic Toda systems are best interpreted as systems on the regular triangular lattice. A general theory of Toda-type systems on graphs and their relation to quad-graph equations has been developed in [6], [3], [9]. A blend of both approaches, one- and two-dimensional, turns out to be fruitful for both ones.

Another point we plan to investigate is the quantum counterpart of the results presented here. It is well known that Bäcklund transformations for the standard Toda lattice admit a natural quantum analog, the Baxter's Q-operator [16]. The Lagrangian of the Bäcklund transformation is a quasi-classical limit of the kernel of the integral Q-operator. The spectrality property of the Bäcklund transformation is the quasi-classical limit of the Baxter's equation relating the monodromy matrix and the Q-operator [17]. At the same time, the quantum counterpart of the whole multi-time Lagrangian theory, in particular, of the closure relation, is not yet clear. Also here, the two-dimensional point of view will be fruitful, as indicated by the work [5] treating a solvable model of statistical mechanics, for which the thermodynamical limit of the partition function is nothing but the action functional of a certain discrete Toda-type model. In this framework, the closure relation obtains its interpretation as the thermodynamical limit of the Z -invariance of the partition function. A blend of both approaches will likely deliver a rather universal and simple picture.

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